MM3 Response of Dynamic Systems

Readings:

- Section 3.3 (response & pole locations, p.118-126);
- Section 3.4 (time-domain specifications, p.126-131)
- Section 3.6 (numerical simulation, p.138-143)

What have we talked in **MM2**?

- ODE models
- Laplace transform
- Block diagram transformation

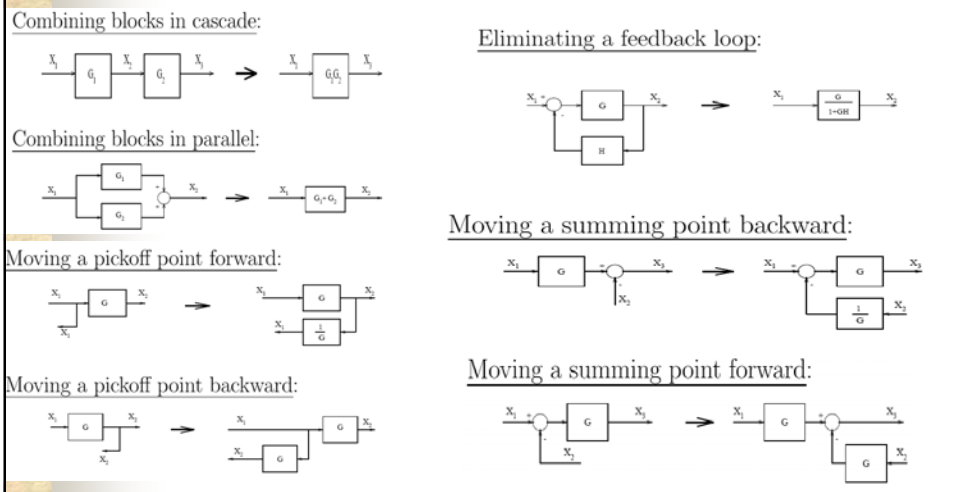
MM2: ODE Model

- A general ODE model:

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0\, y(t) = b_m \frac{d^m u}{dt^m} + \cdots + b_1 \frac{du}{dt} + b_0\, u$$

- **SISO, SIMO, MISO, MIMO** models
- Linear system, Time-invance, Linear Time-Invarance (**LTI**)
- Solution of ODE is an explicit description of dynamic behavior
- Conditions for unique solution of an ODE
- Solving an ODE:
  - Time-domain method, e.g., using exponential function
  - Complex-domain method (**Laplace transform**)
  - Numerical solution – CAD methods, e.g., ode23/ode45
-

MM2: Block diagram Rules



MM2: Simulink Block diagram

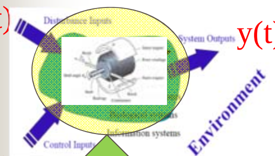
- System build-up
  - Using TF block
  - Using nonlinear blocks
  - Using math blocks
- Creat subsystems
  - Top-down
  - Bottom-up
- Usage of ode23 & ode45

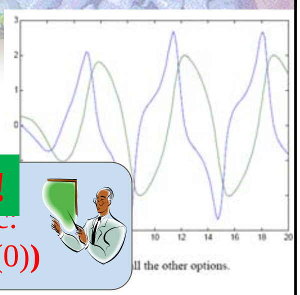
What is the new for today?

Goals for this lecture (MM3)

- Time response analysis
  - Typical inputs
  - 1st, 2nd and higher order systems
- Performance specification of time response
  - Transient performance
  - Steady-state performance
- Numerical simulation of time response

System Models: ODE/TF





Time response!

(explicit) Dynamic:

y(t)=**f(**d(t), u(t), y(0)**)**

$\theta_m(0) = \alpha, \quad \dot\theta_m(0) = \beta$

ODE model

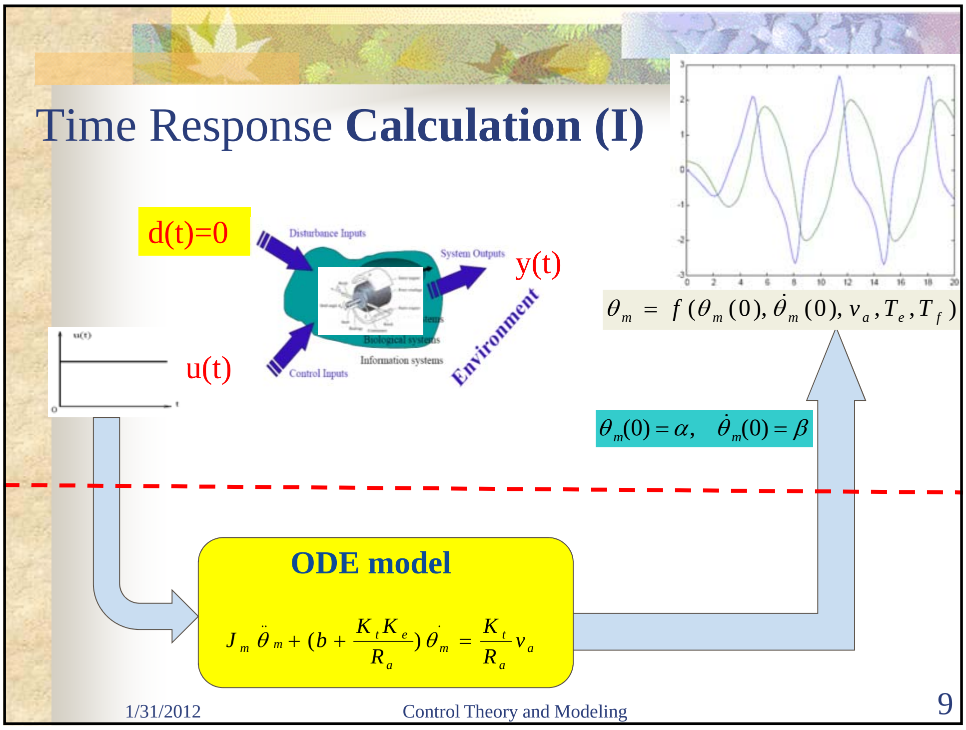
$$J_m \ddot\theta_m + (b + \frac{K_t K_e}{R_a})\dot\theta_m = \frac{K_t}{R_a} v_a - T_e$$

TF model

$$G_1 = \frac{\Theta_m(S)}{V_a} = \frac{\frac{K_t}{R_a}}{J_m s^2 + (b + \frac{K_t K_e}{R_a})s} = \frac{K_t}{s(J_m R_a s + (b + K_t K_e))}$$

$$G_2 = \frac{\Theta_m(S)}{T_e} = \frac{-1}{J_m s^2 + (b + \frac{K_t K_e}{R_a})s} = \frac{-R_a}{s(J_m R_a s + (b + K_t K_e))}$$

System (dynamic) feature description


Time Response Calculation (I)
d(t)=0
Disturbance Inputs
System Outputs
y(t)
u(t)
Control Inputs
Information systems
Environment
$\theta_m = f(\theta_m(0), \dot{\theta}_m(0), v_a, T_e, T_f)$
$\theta_m(0) = \alpha, \quad \dot{\theta}_m(0) = \beta$
ODE model
$J_m \ddot{\theta}_m + (b + \frac{K_t K_e}{R_a})\dot{\theta}_m = \frac{K_t}{R_a} v_a$
1/31/2012
Control Theory and Modeling
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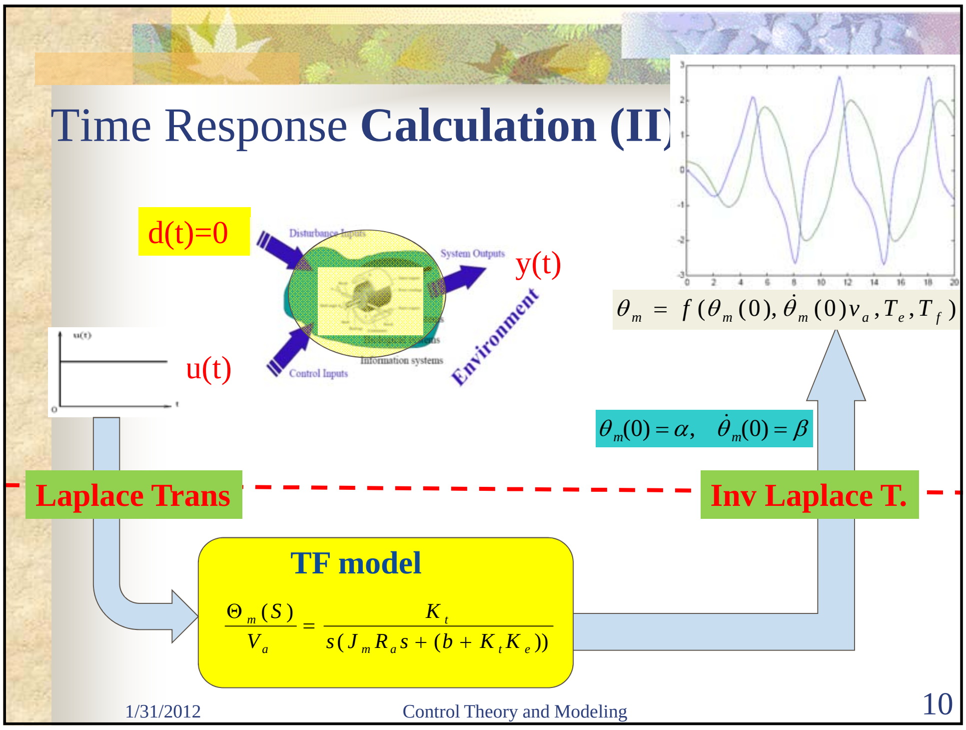
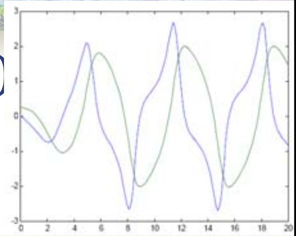
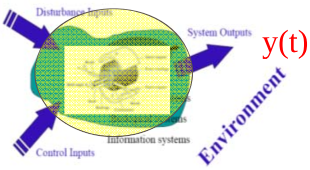

Time Response Calculation (II)
d(t)=0
Disturbance Inputs
System Outputs
y(t)
u(t)
Control Inputs
Information systems
Environment
$\theta_m = f(\theta_m(0), \dot{\theta}_m(0) v_a, T_e, T_f)$
$\theta_m(0) = \alpha, \quad \dot{\theta}_m(0) = \beta$
Laplace Trans
Inv Laplace T.
TF model
$\frac{\Theta_m(S)}{V_a} = \frac{K_t}{s(J_m R_a s + (b + K_t K_e))}$
1/31/2012
Control Theory and Modeling
10

Time Response **Calculation (II)**

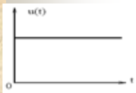


$$\theta_m = f(\theta_m(0), \dot{\theta}_m(0), v_a, T_e, T_f)$$

d(t)=0



y(t)



u(t)

$\theta_m(0) = \alpha, \quad \dot{\theta}_m(0) = \beta$

Laplace Trans

Inv Laplace T.

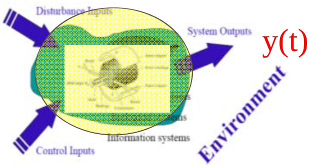
TF model

$$Y(s) = G(s)U(s)$$

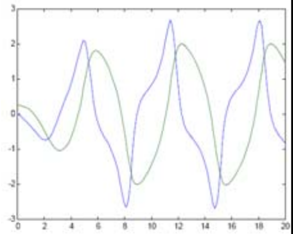
$$\Theta_m(S) = G(s)V_a(s)$$

Time Response **Calculation (III)**

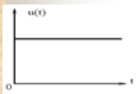
d(t)=0



y(t)



"help ode23" to find out all the other options.



u(t)

$\theta_m(0) = \alpha, \quad \dot{\theta}_m(0) = \beta$

TF model in Matlab
Num= [K];
Den=[JR b+ktKf 0];
lsim(tf(Num,Den),u, T,X0)

$$\frac{\Theta_m(S)}{V_a} = \frac{K_t}{s(J_m R_a s + (b + K_t K_e))}$$

Time Response Analysis

- **Objective:**

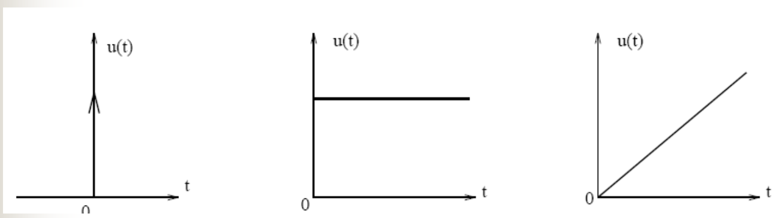
  Based on TF model, Can we predict some **key features** of a dynamic response regarding to some **typical input**, without detail calculation of the solution? In another word, to get a rough sketch of the dynamic with all key (concerned) features kept

- **Typical inputs**
- **Corresponding responses**
- **Key features**

Time Response Analysis – Typical Inputs

- Supercomposition principle
- LTI system
- Typical representations

Test signal	$u(t)$	$U(s)$
Impulse	$\delta(t)$	1
Step	1	$1/s$
Ramp	t	$1/s^2$



Time Response Analysis – **Impulse Signal**

- Impulse signal
  - Features
  - Convolution integration
  - Approximation

Impulse	$\delta(t)$	1

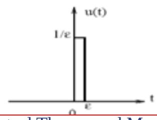
$$\int_{0^-}^{0^+} \delta(t)dt = 1$$

$$f(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t) * \delta(t)$$

Approximation of impulse by a rectangular function

$$\Delta_\varepsilon(t) = \begin{cases} 1/\varepsilon, & 0 \le t \le \varepsilon \\ 0, & \text{otherwise} \end{cases}$$

where $\varepsilon > 0$ is sufficiently small

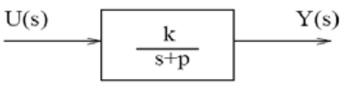


Time Response – **First-order System (I)**

- **First-order** system

Examples: (mm2)

Motor speed control
Cruise control
RC cuircuit
One-tank level control



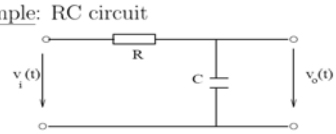
$$G(s) = \frac{k}{s+p}, \quad \text{pole}: \ -p, \quad \text{time constant}: \frac{1}{p}$$

$$G(s) = \frac{c}{\tau s+1}, \quad \text{pole}: \ -\frac{1}{\tau}, \quad \text{time constant}: \tau$$

time domain :

$$g(t) = ke^{-pt} \quad \text{or} \quad g(t) = \frac{c}{\tau}e^{-\frac{1}{\tau}t}$$

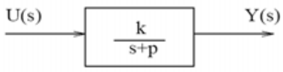
Example: RC circuit



$$G(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{RCs+1}$$

Time Response – **First-order System (II)**

$$\frac{dy}{dt} + py = ku$$



$$(s+p)Y(s) = kU(s) + y(0_-)$$

$$\Rightarrow \quad Y(s) = \frac{k}{s+p}U(s) + \frac{y(0_-)}{s+p}$$

Time response:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)U(s)] + y(0_-)e^{-pt}$$

Separation:

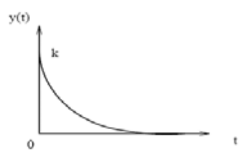
$$\text{Time response} = \begin{matrix}\text{Excitation}\\ \text{response}\end{matrix} + \begin{matrix}\text{Initial condition}\\ \text{response}\end{matrix}$$

First-order System - **Impulse Response**

Time response:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)U(s)] + y(0_-)e^{-pt}$$

Plotting $(p > 0)$:



Input signal: $u(t) = \delta(t) \leftrightarrow U(s) = 1$

Impulse response:

$$y(t) = \mathcal{L}^{-1}[\frac{k}{s+p}] = ke^{-pt}$$

- Initial-condition response $y(0_-)e^{-pt}$
- Impulse response ke^{-pt}

The coefficients of impulse response can be estimated by free-response

Conclusion: Impulse response & initial-condition response are the same type

First-order System - **Step Response**

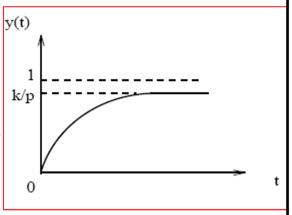
Time response:

$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)U(s)] + y(0_-)e^{-pt}$

Input signal: $u(t) = 1 \leftrightarrow U(s) = 1/s$

$$y(t) = \mathcal{L}^{-1}[\frac{k}{s(s+p)}] = \mathcal{L}^{-1}[\frac{k/p}{s} - \frac{k/p}{s+p}] = \frac{k}{p} - \frac{k}{p}e^{-pt}$$

To follow the step input, a controller of the constant gain p/k is needed



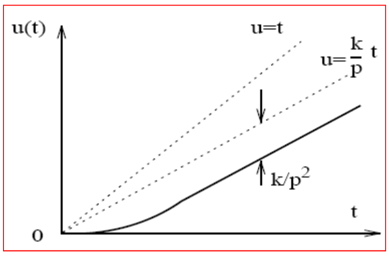
First-order System - **Ramp Response**

Time response:

$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)U(s)] + y(0_-)e^{-pt}$

Input signal: $u(t) = t \leftrightarrow U(s) = 1/s^2$

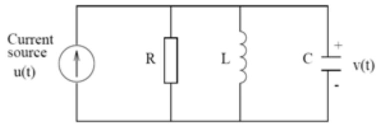
$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[\frac{k}{s^2(s+p)}] \\ &= \mathcal{L}^{-1}[\frac{k/p}{s^2} - \frac{k/p^2}{s} + \frac{k/p^2}{s+p}] \\ &= \frac{k}{p}t - \frac{k}{p^2} + \frac{k}{p^2}e^{-pt} \\ &= \frac{k}{p}t - \frac{k}{p^2}(1 - e^{-pt}) \end{aligned}$$



Type of systems (mm4)

Time Response – **Second-order System (I)**

- **Second-order** system



$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

- ξ – *damping ratio*, a dimensionless factor
- ω_n – *natural frequency* with unit rad/s

Kirchoff's law: $i_R + i_L + i_C = i_s \Rightarrow$
Integro-differential equation

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = u(t)$$

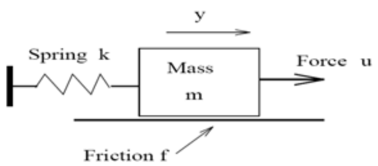
$v(t)$ - voltage of C

$$LC\frac{d^2i(t)}{dt^2} + \frac{L}{R}\frac{di(t)}{dt} + i(t) = u(t)$$

$i(t) = \frac{1}{L}\int_0^t v(t)\,dt$ - current of L

Time Response – **Second-order System (II)**

- **Second-order** system



$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

- ξ – *damping ratio*, a dimensionless factor
- ω_n – *natural frequency* with unit rad/s

Newton's law: $m\,a = F \Rightarrow$

$$m\frac{d^2y(t)}{dt^2} = u(t) - k\,y(t) - f\frac{dy(t)}{dt}$$

$y(t)$ - displacement

$$m\frac{d^2y(t)}{dt^2} + f\frac{dy(t)}{dt} + k\,y(t) = u(t)$$

Second-order System – **Pole (Root) Analysis**

- **Second-order** system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

- ξ – *damping ratio*, a dimensionless factor
- ω_n – *natural frequency* with unit rad/s

Roots of $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$

- Case 1 ($\xi = 1$): Two repeated roots $s_{1,2} = -\omega_n$
- Case 2 ($\xi < 1$): A complex conjugate root pair $s_{1,2} = -\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$
- Case 3 ($\xi > 1$): Two distinct roots $s_{1,2} = -\omega_n(\xi \pm \sqrt{\xi^2 - 1})$

Check execise one for MM2 – pendulum Model analysis)

Second-order System – **Impulse Response**

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Input signal: $u(t) = \delta(t) \leftrightarrow U(s) = 1$

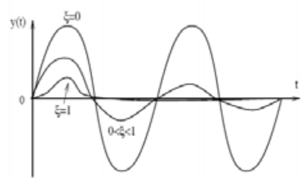
Impulse response: For $\xi < 1$,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[G(s)] \\ &= \mathcal{L}^{-1}\left[\frac{\omega_n^2}{(s+\xi\omega_n)^2 + \omega_d^2}\right] \\ &= \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin \omega_d t \end{aligned}$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$, and for $\xi = 1$

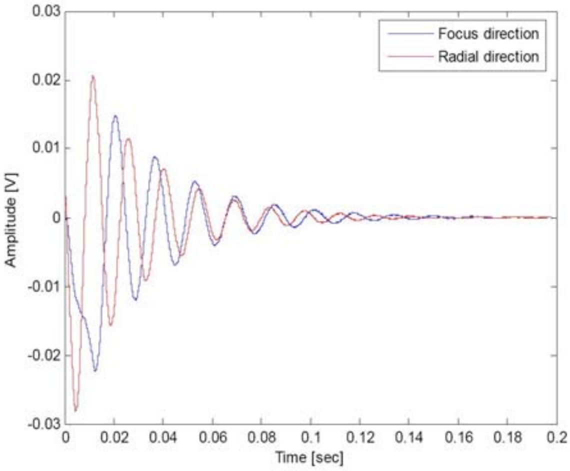
$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{\omega_n^2}{(s+\omega_n)^2}\right] \\ &= \omega_n^2 e^{-\omega_n t} t \end{aligned}$$

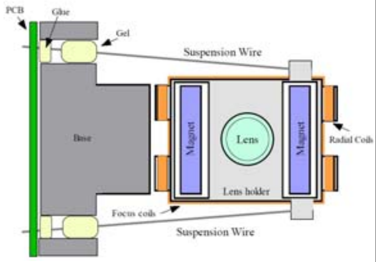
Plotting: For the cases $\xi = 0$, $\xi = 1$ and $0 < \xi < 1$.



The coefficients of impulse response can be estimated by free-response

Example: Impulse Response Estimation





Second-order System – **Step Response (I)**

Input signal: $u(t) = 1 \leftrightarrow U(s) = 1/s$

Case 1: $\xi < 1$

$$Y(s) = \frac{G(s)}{s} = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)}$$
$$= \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$
$$= \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

Inverse Laplace transform $\Rightarrow$

$$y(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\omega_d t$$
$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)$$

with $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$, $\omega_d = \omega\sqrt{1-\xi^2}$.

Remark: Let $c = \sqrt{a^2 + b^2}$, $\alpha = \tan^{-1}\frac{a}{b}$, then

$$a\cos\beta + b\sin\beta = c(\frac{a}{c}\cos\beta + \frac{b}{c}\sin\beta)$$
$$= c(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = c\sin(\alpha + \beta)$$

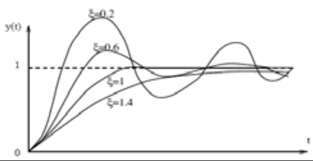
Case 2: $\xi = 1$

$$Y(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2},$$

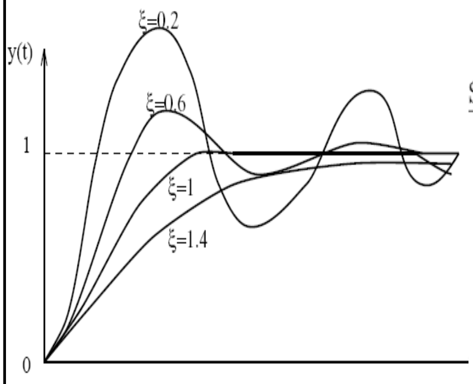
Inverse Laplace transform $\Rightarrow$

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

Plotting:



Second-order System – **Step Response (II)**



Summary: The response

- converges to 1 due to $e^{-\xi\omega_n}$;
- oscillates at frequency ω_d if $0 < \xi < 1$;
- has no oscillation if $\xi \geq 1$;
- possesses larger amplitude of the initial response period for smaller ξ;
- equals $y(t) = 1 - \cos\omega_n t$ for the extreme case: $\xi = 0$

Control specification!

Second-order System – **Ramp Response**

$$Y(s) = \frac{G(s)}{s^2} = \frac{\omega_n^2}{s^2(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$= \frac{1}{s^2} - \frac{2\xi/\omega_n}{s} + \frac{\frac{2\xi}{\omega_n}s + 4\xi^2 - 1}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s^2} - \frac{2\xi/\omega_n}{s} + \frac{2\xi}{\omega_n}\left(\frac{s + \xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} + \frac{\frac{2\xi^2-1}{2\xi}\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}\right)$$

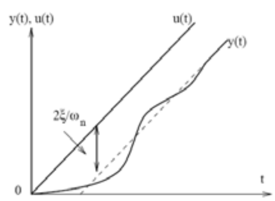
Inverse Laplace transform $\Rightarrow$

$$y(t) = t - \frac{2\xi}{\omega_n} + \frac{2\xi}{\omega_n}e^{-\xi\omega_n t}\left(\cos\omega_d t + \frac{2\xi^2 - 1}{2\xi\sqrt{1-\xi^2}}\sin\omega_d t\right)$$

$$= t - \frac{2\xi}{\omega_n} + \frac{e^{-\xi\omega_n t}}{w_d}\sin(\omega_d t + \phi)$$

with $\phi = \tan^{-1}\frac{2\xi\sqrt{1-\xi^2}}{2\xi^2-1}$, $\omega_d = \omega_n\sqrt{1-\xi^2}$

Plotting:



Conclusion: 2nd-order system can follow a ramp input but with a fixed difference $\frac{2\xi}{\omega_n}$

Second-order System – **Damping Effect**

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Case	Description	Roots
$\xi = 0$	Undampted	$s_{1,2} = \pm j\omega$
$\xi < 1$	Underdampted	$s_{1,2} = -\sigma \pm j\omega$
$\xi = 1$	Critically damped	$s_1 = s_2 = -\sigma$
$\xi > 1$	Overdamped	$s_1 = -\sigma_1,\ s_2 = -\sigma_2$

Time Response – **High-Order System**

Fact: Time response of a higher-order system '=' combination of times responses of 1st- & 2nd-order systems

Method: Partial fraction + inverse Laplace transform

See page 32-37 of the extra readings for detail explanation

Goals for this lecture (MM3)

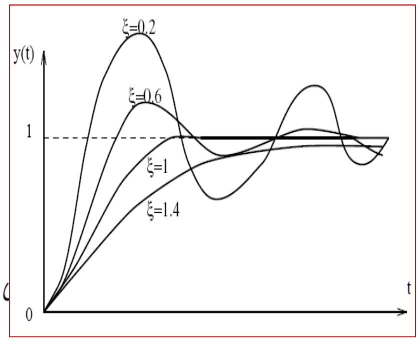
- Time response analysis
  - Typical inputs
  - 1st, 2nd and higher order systems
- **Performance specification of time response**
  - **Transient performance**
  - **Steady-state performance**
- Numerical simulation of time response

Performance Specification

- **Objective:** evaluation of control design
- **Platform:** based on step response of a typical 2nd-order system

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}, \;($$

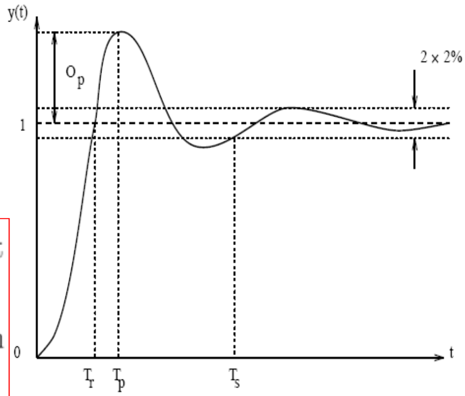


$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Specification – **Transient Performance (I)**

(See FC 126-131...)

- Rise time t_r
- Settling time t_s
- Overshoot M_p
- Peak time t_p



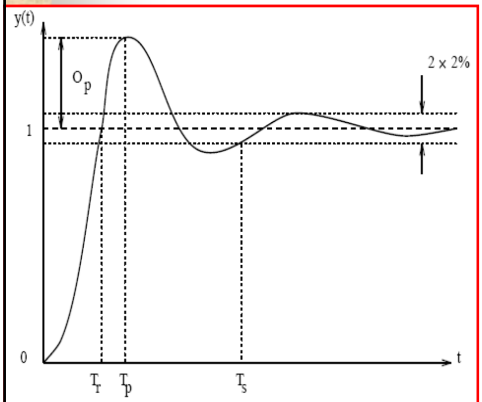
Rise time T_r: taken for the waveform to first reach the final value

Peak time T_p: taken to reach the maximum peak

Settling time T_s: required for the waveform stay within $\pm 2\%$ bound of the final value

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Specification – **Transient Performance (II)**



$$t_r \cong \frac{1.8}{\omega_n}$$

$$t_s \cong \frac{4.6}{\zeta\omega_n} \cong \frac{4.6}{\sigma}$$

$$M_p \cong \begin{cases} 5\%, & \zeta = 0.7 \\ 16\%, & \zeta = 0.5 \\ 35\%, & \zeta = 0.3 \end{cases}$$

$$t_p \cong \frac{\pi}{\omega_d}, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Rise time $T_r = \dfrac{\pi - \phi}{\omega_d}$

Peak time $T_p = \dfrac{\pi}{\omega_d}$

Settling time $T_s \approx \dfrac{4}{\xi\omega_n}$

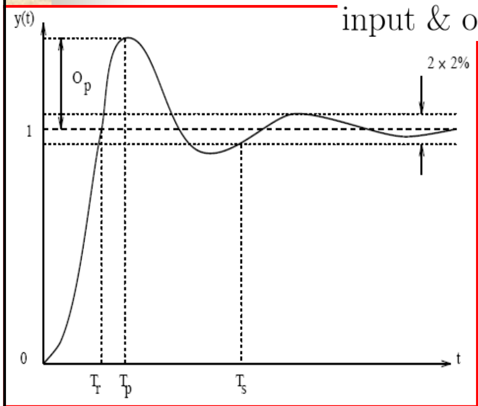
Overshoot $O_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$

$$\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}, \quad \omega_d = \omega\sqrt{1-\xi^2}$$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Specification – **Steady-state Performance**

Steady-state error e_{ss}: difference between input & output as $t \to \infty$



Steady-state error $e_{ss} = 0$

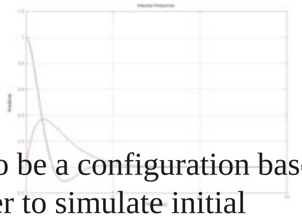
$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

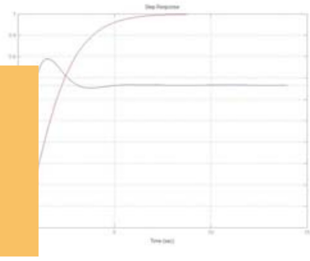
Goals for this lecture (MM3)

- Time response analysis
  - Typical inputs
  - 1st, 2nd and higher order systems
- Performance specification of time response
  - Transient performance
  - Steady-state performance
- **Numerical simulation of time response**

Numerical Simulation of Time Response

- Impulse response: **impulse(sys)**
- Step response: **step(sys)**
- **ltiview(sys)**
- Decomposition of transfer function to be a configuration based on integor and gain elements (in order to simulate initial response in simulink)





EXAMPLE:

```
sys1:                Sys2:
num1=[1];            num2=[1 2];
den1=[1 2 1];        den2=[1 2 3];
impulse(tf(num1,den1),'r',tf(num2,den2),'b')
step(tf(num1,den1),'r',tf(num2,den2),'b')
```