Control Systems Analysis & Design

Module 13471 (1/2)

Ming Hou

Office: AS1-268; Phone: 5063
E-mail: m.hou@hull.ac.uk

Objective: Introduction of fundamentals in control systems analysis and design based on the transfer function description of linear systems

References

M Driels, Linear Control Systems Engineering. McGraw-Hill, 1995

R C Dorf & R H Bishop, Modern Control Systems. 7th Ed., Addition-Wesley, 1995

J Wilkie et al, Control Engineering: An Introductory Course. Palgrave, 2002.

Contents

- Mathematical Model
- Transfer Function
- Time Response Analysis
- Poles & Stability
- Steady-State Error
- PID Control

Concepts:

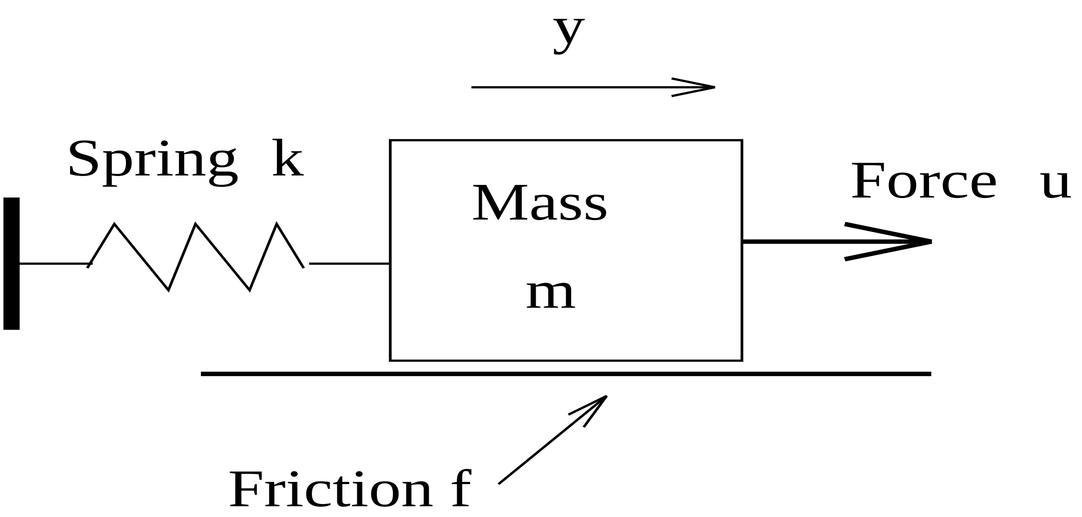
- *System*: A collection of interacting objects
- *Dynamic system*: Non-static interacting
- *Control*: A means of influencing system to have desired behavors

Chapter 1: Mathematical Model

Mathematical model: System description in mathematical form

Remark: Physical system $\leftrightarrow$ Math. model

1.1 Mass-Spring-Damper System



Newton's law: $m\,a = F \Rightarrow$

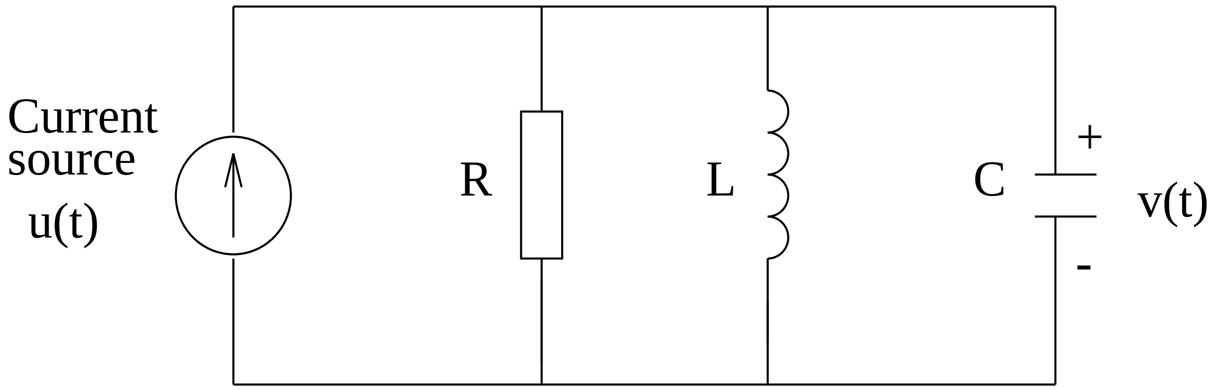
$$m\frac{d^2y(t)}{dt^2} = u(t) - k\,y(t) - f\,\frac{dy(t)}{dt}$$

$y(t)$ - displacement

$\Rightarrow$ Math. model of 2nd-order ODE

$$m\,\frac{d^2y(t)}{dt^2} + f\,\frac{dy(t)}{dt} + k\,y(t) = u(t)$$

1.2 R-L-C Circuit System



Kirchoff's law: $i_R + i_L + i_C = i_s \Rightarrow$
Integro-differential equation

$$\frac{v(t)}{R} + C\,\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)\,dt = u(t)$$

$v(t)$ - voltage of C

$\Rightarrow$ Math. model of 2nd-order ODE

$$LC\,\frac{d^2 i(t)}{dt^2} + \frac{L}{R}\,\frac{di(t)}{dt} + i(t) = u(t)$$

$i(t) = \dfrac{1}{L}\int_0^t v(t)\,dt$ - current of L

<u>Remark</u>: Choosing different variables $\Rightarrow$ different math. descriptions

1.3 Analogy of Dynamic Systems

Facts: M-S-D & R-L-C systems

- are physically completely different
- have the same form of mathematical models

$$a\,\frac{d^2x(t)}{dt^2} + b\,\frac{dx(t)}{dt} + c\,x(t) = u(t)$$

Coefficients	M-S-D system	R-L-C system
a	m	LC
b	f	L/R
c	k	1

Conclusion: M-S-D & R-L-C systems are analogous

Significance: General control theory & method

'Side-effect': Less engineering-looking

1.4 General form of ODEs

$$a_n \frac{d^n x}{dt^n} + \cdots + a_1 \frac{dx}{dt} + a_0 x =$$
$$b_m \frac{d^m u}{dt^m} + \cdots + b_1 \frac{du}{dt} + b_0 u$$

- a_i & b_i - real constants
- n - order of the system
- $n > m$ - reasonable assumption

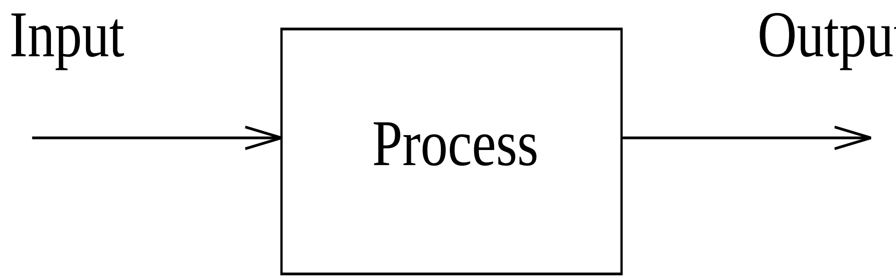
Features

- Linear
- Time-invariant
- Single-input & single-output

Chapter 2: Transfer Function

Transfer function: An input-output mathematical description of a system

2.1 Input-Output Aspect of Systems



Concepts

- *External signals*: Input & output
- *Input*: Active signal, can be specified/designed
- *Output*: Passive signal, can be measured/ monitored

Multi-choices of output:

- *Output for M-S-D system*: y, $\dot{y}$ or $\ddot{y}$
- *Output for R-L-C system*: i_R, i_L or i_C

2.2 Laplace Transform

$$\text{Time function} \quad \overset{\text{Transform}}{\leftrightarrow} \quad \text{Complex function}$$

$(f(t) \leftrightarrow F(s)$, where $s = \sigma + j\omega$, $j = \sqrt{-1})$

Definitions:

- Laplace transform:

$$\mathcal{L}[f(t)] = \int_{0_-}^{\infty} f(t)e^{-st}\,dt = F(s)$$

- Inverse Laplace transform:

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}\,ds = f(t)$$

Example: For $f(t) = t$,

$$\begin{aligned} F(s) &= \int_{0_-}^{\infty} te^{-st}\,dt = -\frac{1}{s}\int_{0_-}^{\infty} t\,de^{-st} \\ &= -\frac{1}{s}\left(te^{-st}|_{0_-}^{\infty} - \int_{0_-}^{\infty} e^{-st}dt\right) \\ &= \frac{1}{s}\int_{0_-}^{\infty} e^{-st}dt = -\frac{1}{s^2}e^{-st}|_{0_-}^{\infty} = \frac{1}{s^2} \end{aligned}$$

Basic Laplace Transform Pairs

Time function $f(t)$	Laplace transform $F(s)$
$\delta(t)$, unit impulse	1
1, unit step	$\frac{1}{s}$
t	$\frac{1}{s^2}$
e^{-at}	$\frac{1}{s+a}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$

Properties of Laplace Transform

(a) $\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha F(s) + \beta G(s)$

(b) $\mathcal{L}^{-1}[\alpha F(s) + \beta G(s)] = \alpha f(t) + \beta g(t)$

(c) $\mathcal{L}[\frac{df(t)}{dt}] = sF(s) - f(0_-)$

(d) $\mathcal{L}[\int_{-\infty}^{t} f(t)\, dt] = \dfrac{F(s)}{s} + \dfrac{1}{s}\int_{-\infty}^{0} f(t)\, dt$

(e) $\mathcal{L}[e^{-at} f(t)] = F(s+a)$

(f) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

2.3 Transfer Function

Motivition: To obtain a ratio of output & input (transfer function), instead of input-output relation (ODE)

Ratio of output & input:

$$\text{Transfer function} \;=\; \frac{\text{Output}}{\text{Input}}$$

Definition:

$$G(s) = \frac{\mathcal{L}[y(t)]}{\mathcal{L}[u(t)]} = \frac{Y(s)}{U(s)}$$

Assumption: Zero initial conditions

Advantages of $G(s)$-description:

- Direct representation of system properties
- Algebraic description

Examples

M-S-D System

Differential equation:

$$m\frac{d^2y(t)}{dt^2} + f\,\frac{dy(t)}{dt} + k\,y(t) = u(t)$$

Laplace transforming both sides $\Rightarrow$

$$(ms^2 + fs + k)Y(s) = U(s)$$

Transfer function $\Rightarrow$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{ms^2 + fs + k}$$

Remarks:

- Formula:
$$\mathcal{L}[\frac{d^2f(t)}{dt^2}] = s^2F(s) - f(0_-)s - \dot{f}(0_-)$$
- Zero initial condition:
$$f(0_-) = 0, \quad \dot{f}(0_-) = 0$$
- Differential operator:
$$\frac{d^i}{dt^i} \leftrightarrow s^i$$

R-L-C System

Case 1: Integral-differential equation:

$$\frac{v(t)}{R} + C\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)dt = u(t)$$

Laplace transforming both sides ⇒

$$(\frac{1}{R} + Cs + \frac{1}{Ls})V(s) = U(s)$$

Transfer function ⇒

$$G(s) = \frac{V(s)}{U(s)} = \frac{1}{\frac{1}{R} + Cs + \frac{1}{Ls}} = \frac{RLs}{RLCs^2 + Ls + R}$$

Remark: Integral operator:

$$\int \cdots \int \leftrightarrow 1/s^i$$

Case 2: Differential equation:

$$LC\frac{i^2(t)}{dt^2} + \frac{L}{R}\frac{di(t)}{dt} + i = u(t)$$

Transfer function ⇒

$$G(s) = \frac{I(s)}{U(s)} = \frac{1}{LCs^2 + \frac{L}{R}s + 1} = \frac{R}{RLCs^2 + Ls + R}$$

Remark: Denominator remains unchanged.

General system description

Differential equation:

$$a_n \frac{d^n y(t)}{dt^n} + \cdots + a_1 \frac{dy(t)}{dt} + a_0\, y(t) = b_m \frac{d^m u(t)}{dt^m} + \cdots + b_1 \frac{du(t)}{dt} + b_0\, u(t)$$

Laplace transforming both sides $\Rightarrow$

$$(a_n s^n + \cdots + a_1 s + a_0) Y(s) = (b_m s^m + \cdots + b_1 s + b_0) U(s)$$

Transfer function $\Rightarrow$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

Remarks:

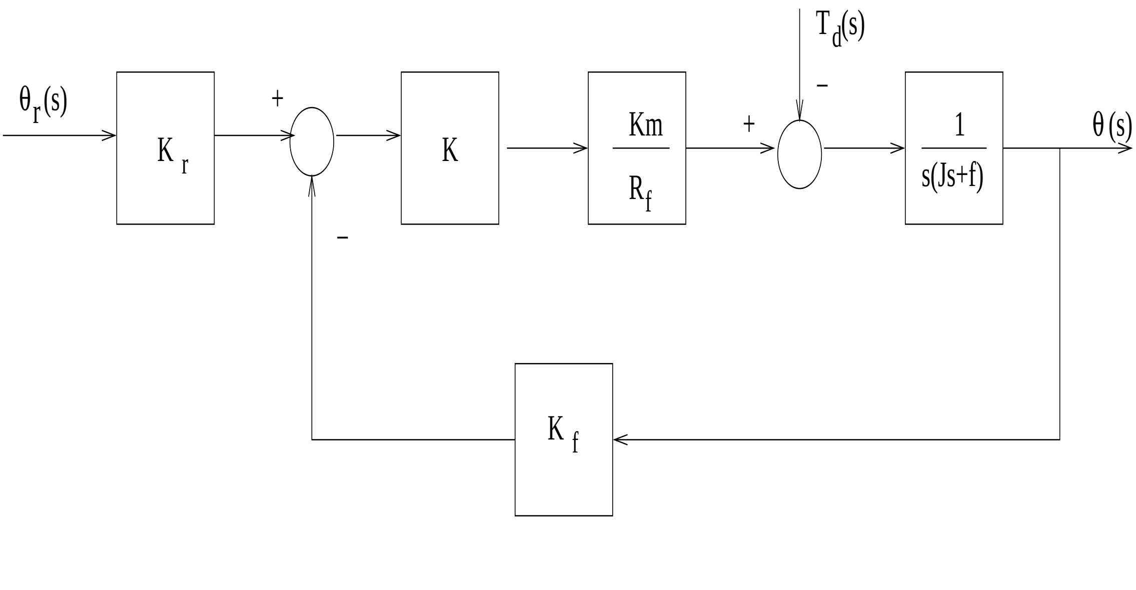
- Formula:
$$\mathcal{L}\left[\frac{d^k f(t)}{dt^k}\right] = s^k F(s) - s^{k-1} f(0_-) - \cdots - s f^{(k-2)}(0_-) - f^{(k-1)}(0_-)$$
- Straightforward extension to integral-differential equation descriptions

2.4 Block Diagram

Objective: To graphically represent a system based on transfer functions of system's components

Example: DC servo-motor system

Physical system's illustration
(Page 10 in the lecturenotes)

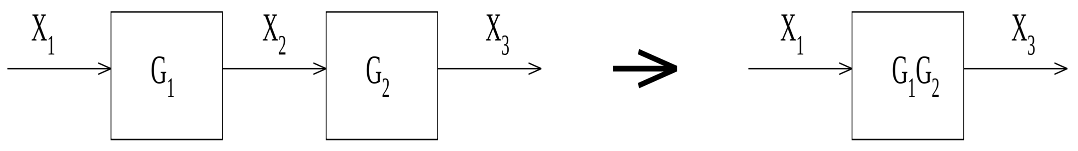


2.4.1 Block Diagram Transformation

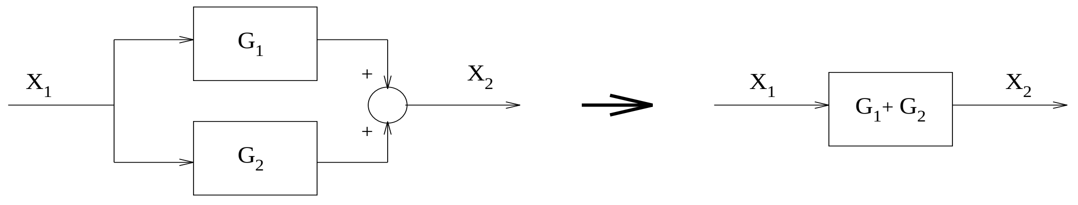
Aim: Composition or decomposition of block diagram

Rules:

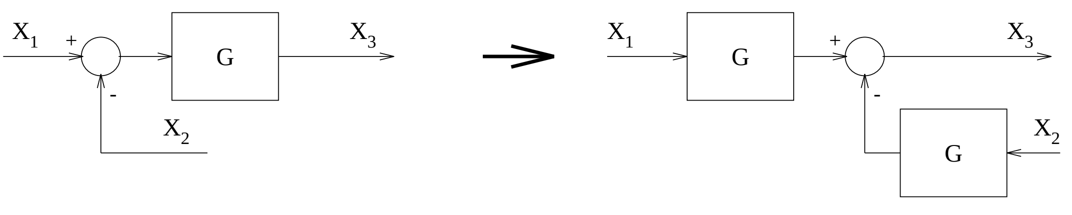
Combining blocks in cascade:



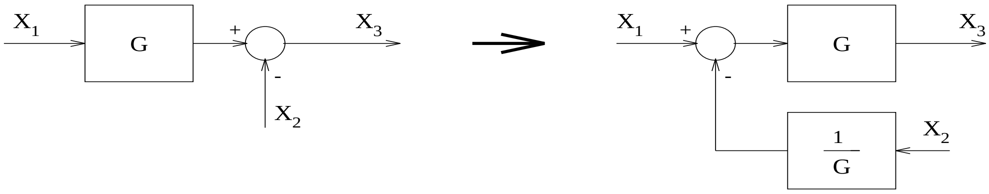
Combining blocks in parallel:



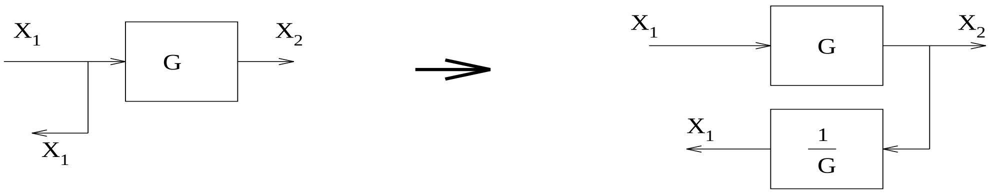
Moving a summing point forward:



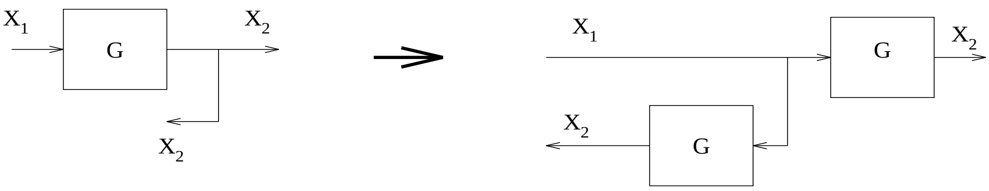
Moving a summing point backward:



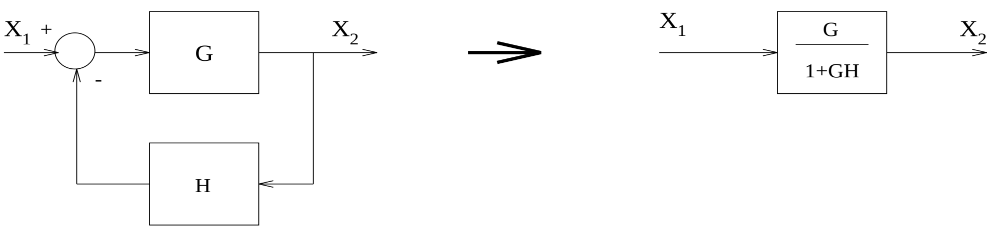
Moving a pickoff point forward:



Moving a pickoff point backward:



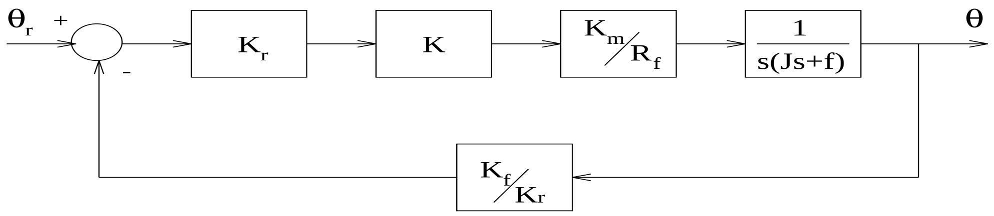
Eliminating a feedback loop:



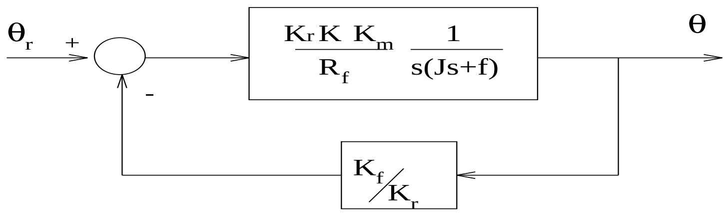
Deriving transfer function for DC motor control system

(a) Moving K_r-block forward
(Moving the summing point backward)

(b) Combing blocks in cascade



(a)



(b)

(c) Eliminating the feedback loop ⇒

$$G(s) = \frac{\theta(s)}{\theta_r(s)} = \frac{K_r K K_m}{R_f J s^2 + R_f f s + K_f K K_m}$$

Chapter 3. Time Response Analysis

Objective: Analyse system's response w.r.t. specific inputs

Method: $y(t) = \mathcal{L}^{-1}[G(s)U(s)]$

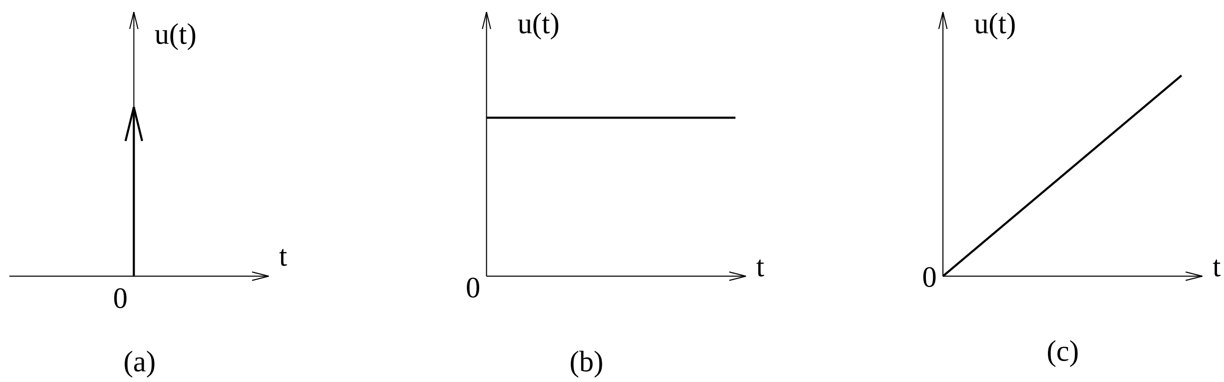
Strategy:

- First: Analysis of 1st- & 2nd-order systems
- Second: Extension to highter-order systems

3.1 Input Signal

Three test signals

Test signal	$u(t)$	$U(s)$
Impulse	$\delta(t)$	1
Step	1	$1/s$
Ramp	t	$1/s^2$

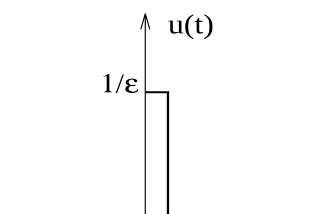


(a) Impulse; (b) Step; (c) Ramp

Approximation of impulse by a rectangular function

$$\Delta_\varepsilon(t) = \begin{cases} 1/\varepsilon, & 0 \le t \le \varepsilon \\ 0, & \text{otherwise} \end{cases}$$

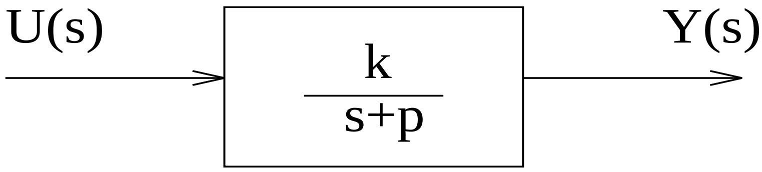
where $\varepsilon > 0$ is sufficiently small



Approximation of impulse

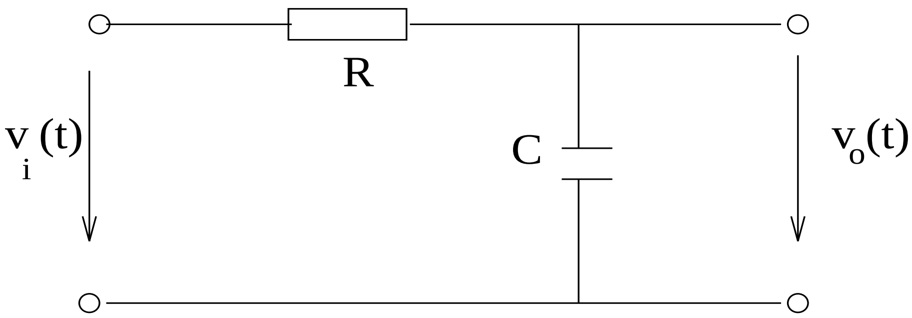
3.2 First-Order Systems

The simplest dynamic systems



Transfer function representation of 1st-order systems

Example: RC circuit



Input-output voltage relation

$$\frac{v_i(t) - v_o(t)}{R} = C\frac{dv_o}{dt}$$

$$\Rightarrow \quad G(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{RCs + 1}.$$

Complete Response

1st-order system $G(s) = \dfrac{k}{s+p}$ in differential equation form

$$\frac{dy}{dt} + py = ku\,.$$

Laplace transform $\Rightarrow$

$$(s+p)Y(s) = kU(s) + y(0_-)$$

$$\Rightarrow \quad Y(s) = \frac{k}{s+p}U(s) + \frac{y(0_-)}{s+p}\,.$$

Time response:

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}[G(s)U(s)] + y(0_-)e^{-pt}$$

Separation:

$$\text{Time response} = \begin{array}{c}\text{Excitation}\\ \text{response}\end{array} + \begin{array}{c}\text{Initial condition}\\ \text{response}\end{array}$$

Concepts: Assume $p > 0$.

- *transient response*: Initial condition response
- *steady-state response*: Excitation response

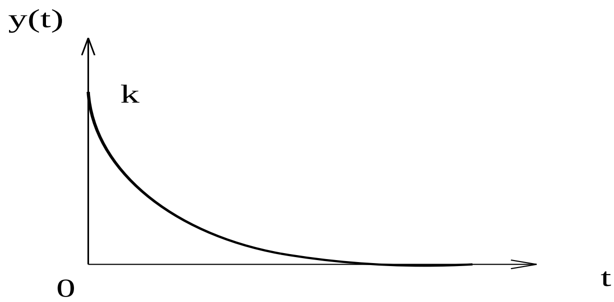
Impulse Response

Input signal: $u(t) = \delta(t) \leftrightarrow U(s) = 1$

Impulse response:

$$y(t) = \mathcal{L}^{-1}[\frac{k}{s+p}] = ke^{-pt}$$

Plotting ($p > 0$):



Comparison:

- Initial-condition response $y(0_-)e^{-pt}$
- Impulse response ke^{-pt}

Conclusion: Impulse response & initial-condition response are the same type

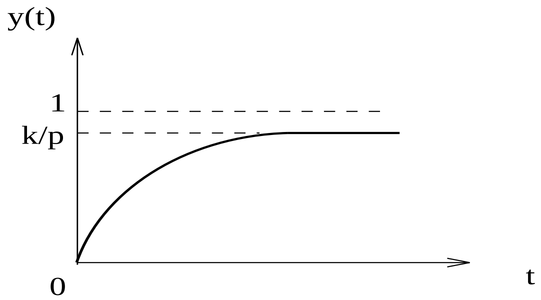
Step Response

Input signal: $u(t) = 1 \leftrightarrow U(s) = 1/s$

Step response:

$$
\begin{aligned}
y(t) &= \mathcal{L}^{-1}[\frac{k}{s(s+p)}] \\
&= \mathcal{L}^{-1}[\frac{k/p}{s} - \frac{k/p}{s+p}] \\
&= \frac{k}{p} - \frac{k}{p}e^{-pt}
\end{aligned}
$$

Plotting



Conclusion: To follow the step input, a controller of the constant gain p/k is needed.

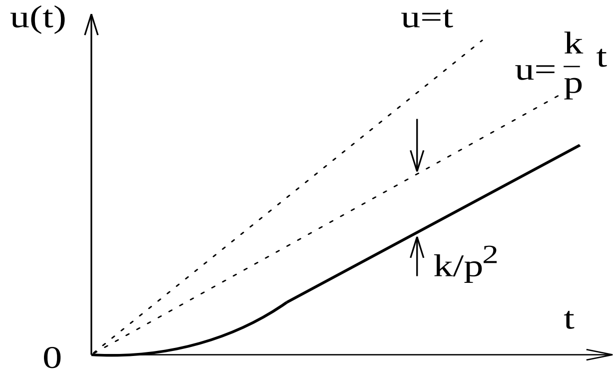
Ramp Response

Input signal: $u(t) = t \leftrightarrow U(s) = 1/s^2$

Ramp response:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[\frac{k}{s^2(s+p)}] \\ &= \mathcal{L}^{-1}[\frac{k/p}{s^2} - \frac{k/p^2}{s} + \frac{k/p^2}{s+p}] \\ &= \frac{k}{p}t - \frac{k}{p^2} + \frac{k}{p^2}e^{-pt} \\ &= \frac{k}{p}t - \frac{k}{p^2}(1 - e^{-pt}) \end{aligned}$$

Plotting:



Conclusion: By adding a controller with the constant gain p/k, the 1st-order system can follow the ramp input, but with a time delay $1/p$.

3.3 2nd-Order Systems

Examples: M-S-D & R-L-C systems

Consider: A class of 2nd-order systems described by the transfer function

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

- ξ – ***damping ratio***, a dimensionless factor
- ω_n – ***natural frequency*** with unit rad/s

Values of ξ & ω_n for MSD & RLC systems

System	ω_n	ξ
Mass-spring-damper	$\sqrt{\frac{k}{m}}$	$\frac{f}{2}\frac{1}{\sqrt{mk}}$
RLC circuit	$\frac{1}{\sqrt{LC}}$	$\frac{1}{2R}\sqrt{\frac{L}{C}}$

Root Analysis

Set the denominator to zero $\Rightarrow$

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

Rewriting $\Rightarrow$

$$(s+\xi\omega_n)^2+(\omega_n\sqrt{1-\xi^2})^2=0\,.$$

Roots of $s^2+2\xi\omega_n s+\omega_n^2=0$

- Case 1 ($\xi=1$): Two repeated roots $s_{1,2}=-\omega_n$
- Case 2 ($\xi<1$): A complex conjugate root pair $s_{1,2}=-\xi\omega_n\pm j\omega_n\sqrt{1-\xi^2}$
- Case 3 ($\xi>1$): Two distinct roots $s_{1,2}=-\omega_n(\xi\pm\sqrt{\xi^2-1})$

Explanation of case 3

$$\begin{aligned} G(s) &= \frac{\omega_n^2}{(s+\xi\omega_n)^2-(\omega_n\sqrt{\xi^2-1})^2} \\ &= \frac{2\omega_n}{\sqrt{\xi^2-1}}\Bigg(\frac{1}{s+\omega_n(\xi-\sqrt{\xi^2-1})}- \\ &\qquad \frac{1}{s+\omega_n(\xi+\sqrt{\xi^2-1})}\Bigg)\,. \end{aligned}$$

Hence, analysis of first-order systems can be applied.

Impulse Response

Input signal: $u(t) = \delta(t) \leftrightarrow U(s) = 1$

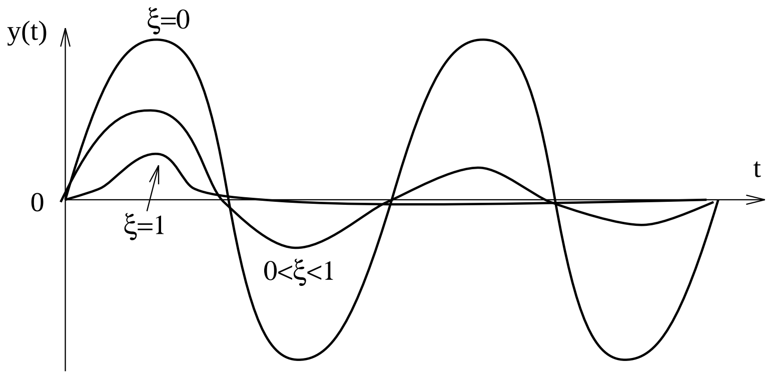
Impulse response: For $\xi < 1$,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[G(s)] \\ &= \mathcal{L}^{-1}\left[\frac{\omega_n^2}{(s+\xi\omega_n)^2+\omega_d^2}\right] \\ &= \frac{\omega_n}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin \omega_d t \end{aligned}$$

where $\omega_d = \omega_n\sqrt{1-\xi^2}$, and for $\xi = 1$

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left[\frac{\omega_n^2}{(s+\omega_n)^2}\right] \\ &= \omega_n^2 e^{-\omega_n t} t \end{aligned}$$

Plotting: For the cases $\xi = 0$, $\xi = 1$ and $0 < \xi < 1$.



Step Response

Input signal: $u(t) = 1 \leftrightarrow U(s) = 1/s$

Case 1: $\xi < 1$

$$
\begin{aligned}
Y(s) &= \frac{G(s)}{s} = \frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} \\
&= \frac{1}{s} - \frac{s + 2\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2} \\
&= \frac{1}{s} - \frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}
\end{aligned}
$$

Inverse Laplace transform $\Rightarrow$

$$
\begin{aligned}
y(t) &= 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t}\sin\omega_d t \\
&= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)
\end{aligned}
$$

with $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$, $\omega_d = \omega\sqrt{1-\xi^2}$.

Remark: Let $c = \sqrt{a^2 + b^2}$, $\alpha = \tan^{-1}\frac{a}{b}$, then

$$
a\cos\beta + b\sin\beta = c(\frac{a}{c}\cos\beta + \frac{b}{c}\sin\beta)
$$

$$
= c(\sin\alpha\cos\beta + \cos\alpha\sin\beta) = c\sin(\alpha + \beta)
$$

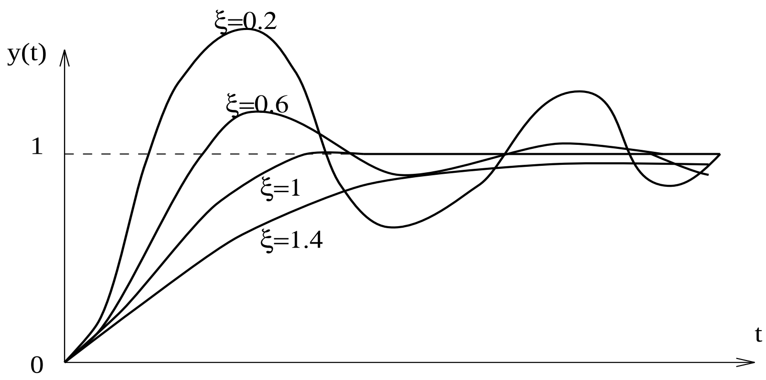
Case 2: $\xi = 1$

$$Y(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2},$$

Inverse Laplace transform $\Rightarrow$

$$y(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

Plotting:



Summary: The response

- converges to 1 due to $e^{-\xi\omega_n}$;
- oscillates at frequency ω_d if $0 < \xi < 1$;
- has no oscillation if $\xi \geq 1$;
- possesses larger amplitude of the initial response period for smaller ξ;
- equals $y(t) = 1 - \cos\omega_n t$ for the extreme case: $\xi = 0$

Ramp Response

Fraction form of $Y(s)$:

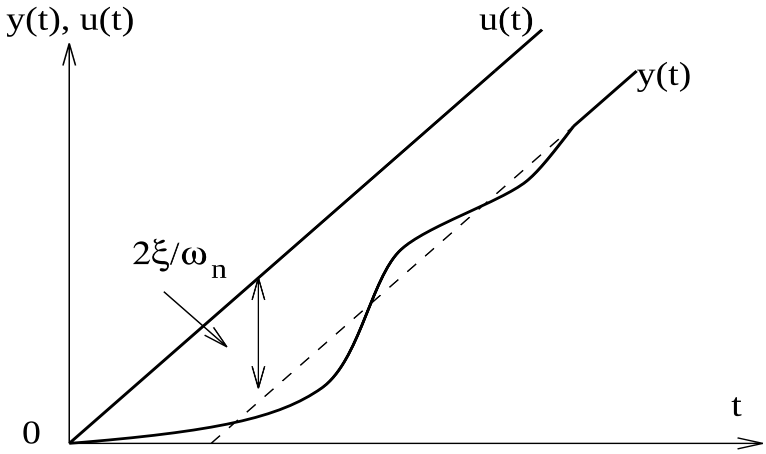
$$\begin{aligned}
Y(s) &= \frac{G(s)}{s^2} = \frac{\omega_n^2}{s^2(s^2+2\xi\omega_n s+\omega_n^2)} \\
&= \frac{1}{s^2} - \frac{2\xi/\omega_n}{s} + \frac{\frac{2\xi}{\omega_n}s + 4\xi^2 - 1}{s^2+2\xi\omega_n s+\omega_n^2} \\
&= \frac{1}{s^2} - \frac{2\xi/\omega_n}{s} + \frac{2\xi}{\omega_n}\left(\frac{s+\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2} + \right. \\
&\qquad \left. \frac{\frac{2\xi^2-1}{2\xi}\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}\right)
\end{aligned}$$

Inverse Laplace transform $\Rightarrow$

$$\begin{aligned}
y(t) &= t - \frac{2\xi}{\omega_n} + \frac{2\xi}{\omega_n}e^{-\xi\omega_n t}\left(\cos\omega_d t + \right. \\
&\qquad \left. \frac{2\xi^2-1}{2\xi\sqrt{1-\xi^2}}\sin\omega_d t\right) \\
&= t - \frac{2\xi}{\omega_n} + \frac{e^{-\xi\omega_n t}}{w_d}\sin(\omega_d t + \phi)
\end{aligned}$$

with $\phi = \tan^{-1}\frac{2\xi\sqrt{1-\xi^2}}{2\xi^2-1}$, $\omega_d = \omega_n\sqrt{1-\xi^2}$

Plotting:



Conclusion: 2nd-order system can follow a ramp input but with a fixed difference $\dfrac{2\xi}{\omega_n}$

Damping Effect

Case	Description	Roots
$\xi = 0$	Undampted	$s_{1,2} = \pm j\omega$
$\xi < 1$	Underdampted	$s_{1,2} = -\sigma \pm j\omega$
$\xi = 1$	Critically damped	$s_1 = s_2 = -\sigma$
$\xi > 1$	Overdamped	$s_1 = -\sigma_1,\ s_2 = -\sigma_2$

3.4 Higher-Order System

Example: Step response for a 3rd-order system

Transfer function

$$G(s) = \frac{2s+1}{s^3+2s^2+2s+1}$$

Note: $s^3+2s^2+2s+1 = (s+1)(s^2+s+1)$

Fraction expression:

$$\begin{aligned} Y(s) &= G(s)U(s) = \frac{G(s)}{s} \\ &= \frac{2s+1}{s(s+1)(s^2+s+1)} \\ &= \frac{a}{s} + \frac{b}{s+1} + \frac{cs+d}{s^2+s+1} \end{aligned}$$

Determination of coefficients a & b

- $a = s\frac{G(s)}{s}|_{s=0} = 1$
- $b = (s+1)\frac{G(s)}{s}|_{s=-1} = 1$

Determination of coefficients c & d

$$\begin{aligned} & \frac{1}{s} + \frac{1}{s+1} + \frac{cs+d}{s^2+s+1} \\ = \; & \frac{(c+2)s^3 + (c+d+3)s^2 + (3+d)s + 1}{s(s+1)(s^2+s+1)} \end{aligned}$$

which implies

$$\begin{cases} c+2=0 \\ c+d+3=0 \\ 3+d=2 \end{cases} \quad \Rightarrow \quad \begin{cases} c=-2 \\ d=-1 \end{cases}$$

Final fraction expression:

$$\begin{aligned} Y(s) &= \frac{1}{s} + \frac{1}{s+1} - \frac{2s+1}{s^2+s+1} \\ &= \frac{1}{s} + \frac{1}{s+1} - 2\frac{s+1/2}{(s+1/2)^2 + (\sqrt{3}/2)^2} \end{aligned}$$

Step response

$$y(t) = 1 + e^{-t} - 2e^{-t/2} \cos \frac{\sqrt{3}}{2} t$$

Fact: Time response of a higher-order system '=' combination of times responses of 1st- & 2nd-order systems

Method: Partial fraction + inverse Laplace transform

A general form of transfer function

$$G(s) = \frac{p(s)}{q(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

Factorisation of $q(s)$

$$q(s) = p_0(s + p_1)(s + p_2) \cdots (s + p_n)$$

where $-p_1, \cdots, -p_n$ – n roots of $q(s) = 0$

Case 1: Distinct Real Roots

Partial fraction expression:

$$\frac{p(s)}{q(s)} = \frac{c_1}{s + p_1} + \frac{c_2}{s + p_2} + \cdots + \frac{c_n}{s + p_n}$$

where $c_i = (s + p_i)\frac{p(s)}{q(s)}\Big|_{s=-p_i}$

Impulse response

$$y(t) = c_1 e^{-p_1 t} + c_2 e^{-p_2 t} + \cdots + c_n e^{-p_n t}$$

Case 2: Repeated Real Roots

Partial fraction expression:

$$\frac{p(s)}{q(s)} = \frac{d_1}{s+p_1} + \frac{d_2}{(s+p_1)^2} + \cdots + \frac{d_n}{(s+p_1)^n}$$

where d_i – determined by comparing coefficients

Impulse response

$$y(t) = e^{-p_1 t}(d_1 + d_2 t + \cdots + d_n t^{n-1})$$

Case 3: Distinct Complex Root Pairs

Conjugate pair: $(s+\sigma+j\omega)(s+\sigma-j\omega) = (s+\sigma)^2+\omega^2$

Partial fraction expression for $\frac{p(s)}{q(s)}$:

$$h_1 \frac{s+e_1}{(s+\sigma_1)^2+\omega_1^2} + \cdots + h_{n/2}\frac{s+e_{n/2}}{(s+\sigma_{n/2})^2+\omega_{n/2}^2}$$

where h_i, e_i, σ_i & ω_i – real coefficients

Impulse response

$$\begin{aligned} y(t) = {} & \bar{h}_1 e^{-\sigma_1 t}\sin(\omega_1 t + \phi_1) + \cdots + \\ & \bar{h}_{n/2} e^{-\sigma_{n/2} t}\sin(\omega_{n/2} t + \phi_{n/2}) \end{aligned}$$

Case 4: Repeated Complex Root Pairs

Partial fraction expression for $\frac{p(s)}{q(s)}$:

$$f_1 \frac{s+g_1}{(s+\sigma)^2+\omega^2} + \cdots + f_{n/2} \frac{s+g_{n/2}}{\left[(s+\sigma)^2+\omega^2\right]^{n/2}}$$

Impulse response:

$$\begin{aligned} y(t) \; = \; & e^{-\sigma t}\left[\bar{f}_1 \sin(\omega t + \bar{\phi}_1) + \cdots + \right. \\ & \left. \bar{f}_{n/2} t^{n/2-1} \sin(\omega t + \bar{\phi}_{n/2})\right] \end{aligned}$$

Case 5: General Case

$$\begin{aligned} q(s) \; = \; & (s+p_1)^{n_1}(s+p_2)^{n_2} \cdots (s+p_k)^{n_k} \cdot \\ & \left[(s+\sigma_1)^2+\omega_1^2\right]^{n_{k+1}} \cdots \left[(s+\sigma_l)^2+\omega_l^2\right]^{n_{k+l}} \end{aligned}$$

where $n_1 + \cdots + n_k + 2(n_{k+1} + \cdots + n_{k+l}) = n$

Partial fraction expression for $\frac{p(s)}{q(s)}$:

$$\sum_{i=1}^{k} \left[\frac{d_{i1}}{s+p_i} + \frac{d_{i2}}{(s+p_i)^2} + \cdots + \frac{d_{in_i}}{(s+p_i)^{n_i}}\right] + \sum_{i=1}^{l}$$

$$\left[f_{i1} \frac{s+g_{i1}}{(s+\sigma_i)^2+\omega_i^2} + \cdots + f_{in_{k+i}} \frac{s+g_{in_{k+i}}}{\left[(s+\sigma_i)^2+\omega_i^2\right]^{n_{k+i}}}\right.$$

Impulse response:

$$y(t) = \sum_{i=1}^{k} e^{-p_i t}(d_{i1}+d_{i2}t+\cdots+d_{in_i}t^{n_i-1})+\sum_{i=1}^{l} e^{-\sigma_i t}$$
$$\left[\bar{f}_{i1}\sin(\omega_i t+\phi_{i1})+\cdots+\bar{f}_{in_{k+i}}t^{n_{k+i}-1}\sin(\omega_i t+\phi_{in_{k+i}})\right]$$

Initial Condition Response

Laplace transform:

$$\mathcal{L}\left[\frac{dy^k}{dt}\right] = Y(s)s^k - y(0_-)s^{k-1} - \cdots - y^{(k-1)}(0_-),$$

$\Rightarrow$

$$a_n \frac{d^n y}{dt^n} + \cdots + a_0\, y = b_m \frac{d^m u}{dt^m} + \cdots + b_0\, u$$

$\Rightarrow$

$$Y(s) = \frac{\bar{p}(s)}{q(s)} = \frac{\bar{b}_{n-1}s^{n-1} + \cdots + \bar{b}_1 s + \bar{b}_0}{a_n s^n + \cdots + a_1 s + a_0}$$

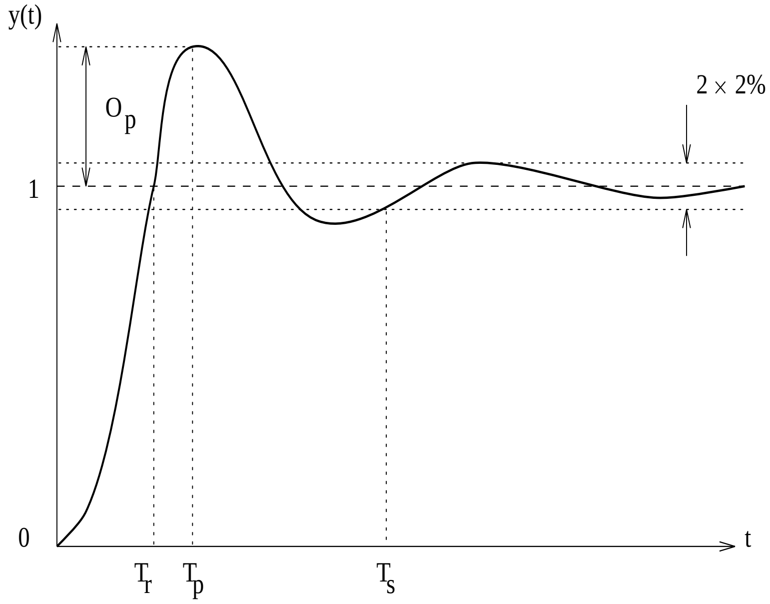
Comparison: Laplace transforms of impulse response $\frac{p(s)}{q(s)}$ & initial-condition response $\frac{\bar{p}(s)}{q(s)}$

Conclusion: Initial-condition response has exactly the same form as the impulse response

3.5 Performance Specification of Time Response

Objective: Evaluation of control designs

Performance specifications: Based on step response of a typical 2nd-order system



- *Rise time* T_r: taken for the waveform to first reach the final value
- *Peak time* T_p: taken to reach the maximum peak
- *Settling time* T_s: required for the waveform stay within $\pm 2\%$ bound of the final value

- *Overshoot* O_p: amount the waveform overshoots the final value at the peak time
- *Steady-state error* e_{ss}: difference between input & output as $t \to \infty$

Summary for 2nd-order systems

- Rise time $T_r = \dfrac{\pi - \phi}{\omega_d}$
- Peak time $T_p = \dfrac{\pi}{\omega_d}$
- Settling time $T_s \approx \dfrac{4}{\xi \omega_n}$
- Overshoot $O_p = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$
- Steady-state error $e_{ss} = 0$

Remarks:

- Non-zero initial conditions
- T_s - approximate solution of $\dfrac{e^{-\xi\omega_n T_s}}{\sqrt{1-\xi^2}} = 2\%$

Chapter 4. Poles & Stability

Stability: A fundamental requirement

4.1 Definition: A system is said to be *stable* if its time response with respect to non-zero initial conditions approaches zero as $t \to \infty$

Unstable: Initial-condition response increases with time

Marginally stable: Initial-condition response remains bounded but does not approach to zero

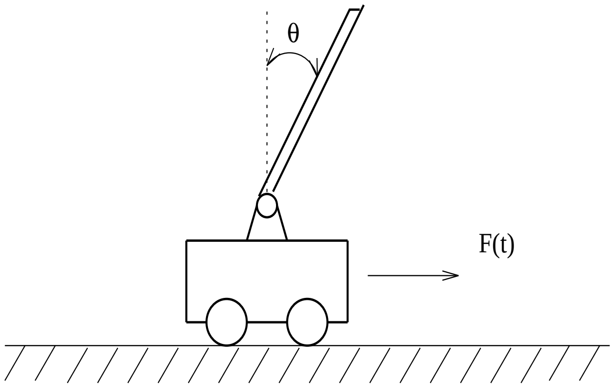
Practical consequences of being not stable:

- For systems with safety measure:
  - output reaches the saturation level
  - automatic shutdown
- For systems without safety measure:
  - mechanical breakdown
  - short circuit
  - blow-up of a chemical reactor

Practical cases:

- Inherently stable (MSD & RLC systems)
- Unstable in nature (Inverted pendulum; ball-beam system)
- Unstable due to error connection or improper controller design

An example: Unstable inverted pendulum



Essential task of control

- To stabilise unstable systems or
- To ensure that stable systems remain stable

4.2 Poles & Zeros

Poles & zeros - Two important concepts defined via the transfer function & related to stability & time response

Polynomial form of $G(s)$:

$$G(s) = \frac{p(s)}{q(s)} = \frac{b_m s^m + \cdots + b_1 s + b_0}{a_n s^n + \cdots + a_1 s + a_0}$$

where a_i and b_i are real numbers

Definitions:

- Poles: Roots of $q(s) = 0$
- Zeros: Roots of $p(s) = 0$
- Characteristic equation: $q(s) = 0$

Pole-Zero Form of $G(s)$

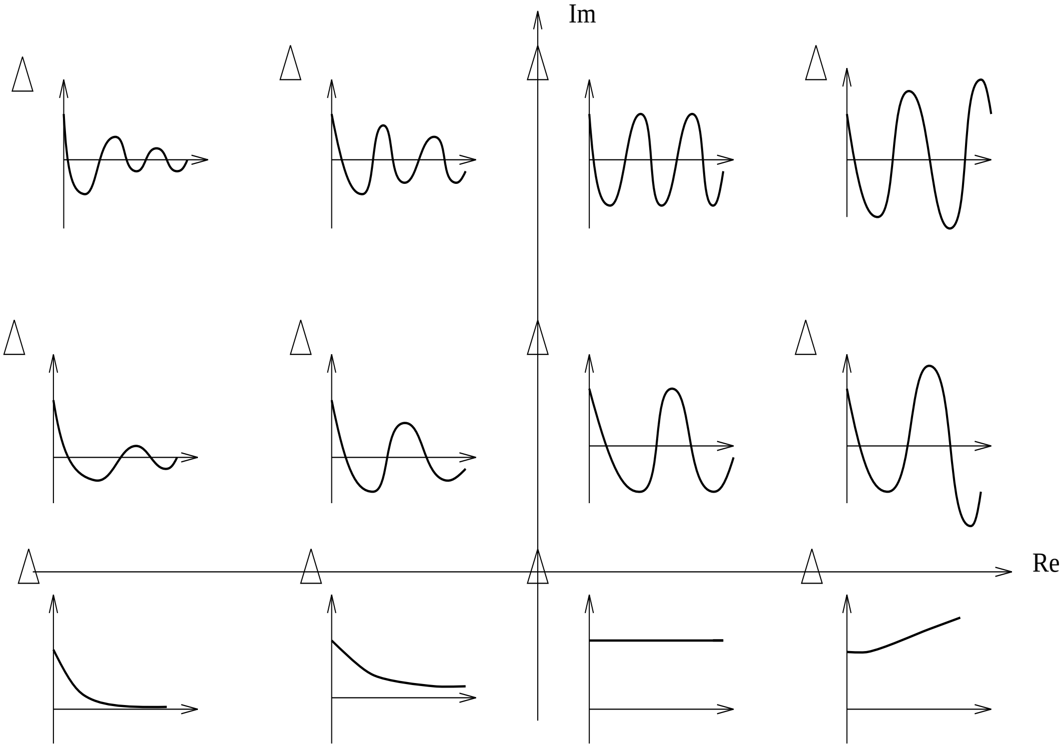
$$G(s) = \frac{p(s)}{q(s)} = k\,\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where $k = b_m/a_n$,

- $-p_i$ – Poles (real numbers/conjugate pairs)
- $-z_i$ – Zeros (real numbers/conjugate pairs)

Effect of Poles

Poles determine types of impulse response



Locations of poles versus types of impulse responses

Effect of Zeros Zeros together with poles determine contribution of each impulse-response type to the complete response

Example: Consider the 3rd-order system

$$G(s) = \frac{2s+1}{s^3+2s^2+2s+1}$$

Zero: $z = -1/2$

Characteristic equation:

$$s^3 + 2s^2 + 2s + 1 = 0$$
$$\Rightarrow (s+1)(s^2+s+1) = 0$$

Poles: $p_1 = -1$, $p_{2,3} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

Fraction expression (types of response)

$$G(s) = \frac{-1}{s+1} + \frac{s+2}{s^2+s+1}$$

Impulse response:

$$y(t) = \mathcal{L}^{-1}[G(s)] = ae^{-t} + be^{-t/2}\sin(\frac{\sqrt{3}}{2}t + \phi)$$

where

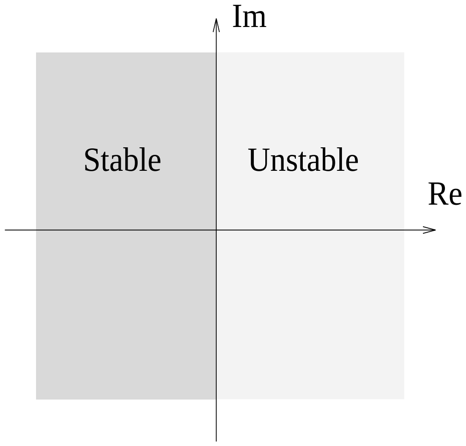
- response types determined by poles
- $a = -1$, $b = 2$ & $\phi = \tan^{-1}\frac{1}{\sqrt{3}}$ determined by poles & zero

4.3 Basic Stability Test

Stability criterion: A linear system is stable if, and only if all its poles have negative real parts

Graphical explanation:

A system is stable if, and only if all the poles are located on the left-hand side of the s-plane



Verification:

- Let $G(s) = \frac{p(s)}{q(s)}$
- Nonzero-initial condition response
  $y(t) = \mathcal{L}^{-1}[\frac{r(s)}{q(s)}]$
- $r(s)$ is a polynomial with coefficents determined by the nonzero initial condition

4.4 Routh-Hurwitz Stability Test

Motivation: Testing stability without calculating poles

Characteristic polynomial $q(s)$:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh-Hurwitz Array

$$\begin{array}{c|llllll}
1 & a_n & a_{n-2} & a_{n-4} & \cdots & \cdots & a_0 \\
2 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & a_1 & \\
3 & b_1 & b_2 & \cdots & \cdots & b_{n/2} & \\
4 & c_1 & c_2 & \cdots & c_{n/2-1} & & \\
\vdots & \vdots & \vdots & & & & \\
n & g_1 & g_2 & & & & \\
n+1 & h_1 & & & & &
\end{array}$$

Constructing first two rows: Up & down, and from left to right

Calculation rule for the remaing entries:

$$\begin{array}{cccc} \times & \times & \times & \times \\ \times & a & b & \times \\ \times & c & d & \times \\ \times & e & \times & \times \end{array} \quad \Rightarrow \quad e = -\frac{1}{c}\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{bc - ad}{c}$$

where '$\times$' – entries of no interest

Routh-Hurwitz Criterion

No changes in sign and no zeros in the 1st column of the array	$\Leftrightarrow$	The system being stable

Routh-Hurwitz criterion: No. of unstable roots of $q(s)$ = No. of changes in sign of the 1st column

Remark

- One/more zeros appearing in the 1st column $\Rightarrow$ poles with zero real part
- Marginally stable if the poles with zero real part are distinct
- Unstable if these poles are repeated

Examples: Stability Test

2nd-order system

$$q(s) = a_2 s^2 + a_1 s + a_0$$

Routh-Hurwitz array

$$\begin{array}{c|cc} 1 & a_2 & a_0 \\ 2 & a_1 & 0 \\ 3 & b_1 & 0 \end{array} \qquad \Rightarrow \qquad \begin{array}{c|cc} 1 & a_2 & a_0 \\ 2 & a_1 & 0 \\ 3 & a_0 & 0 \end{array}$$

as $b_1 = -\dfrac{1}{a_1} \begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = \dfrac{a_1 a_0 - a_2 \cdot 0}{a_1} = a_0$

Conclusion: 2nd-order system is stable $\Leftrightarrow a_2,\ a_1,\ a_0$ have the same sign

3rd-order system

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 \,.$$

Routh-Hurwitz array:

$$\begin{array}{c|cc} 1 & a_3 & a_1 \\ 2 & a_2 & a_0 \\ 3 & b_1 & 0 \\ 4 & c_1 & 0 \end{array} \qquad \Rightarrow \qquad \begin{array}{c|cc} 1 & a_3 & a_1 \\ 2 & a_2 & a_0 \\ 3 & \frac{a_2 a_1 - a_3 a_0}{a_2} & 0 \\ 4 & a_0 & 0 \end{array}$$

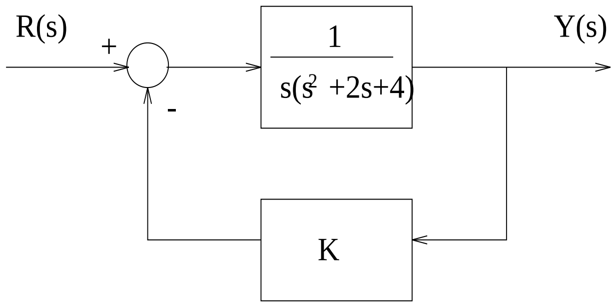
Conclusion: Stability of 3rd-order system $\Leftrightarrow$

(1) a_3, a_2, a_1, a_0 have the same sign;

(2) $a_2 a_1 > a_3 a_0$.

Examples: Controller Design

Example 1: Determine the controller gain K to stablise the 3rd-order system



Characteristic polynomial of controlled system:

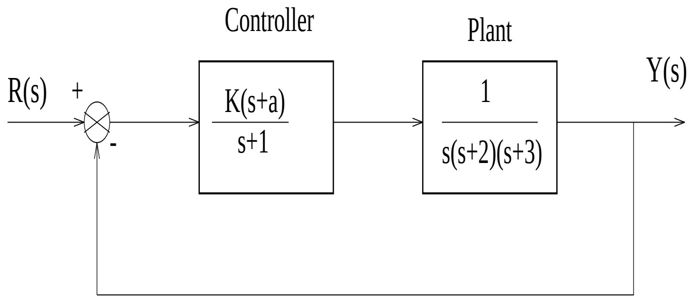
$$s^3 + 2s^2 + 4s + K$$

Routh-Hurwitz array:

$$\begin{array}{c|cc} 1 & 1 & 4 \\ 2 & 2 & K \\ 3 & \frac{8-K}{2} & \\ 4 & K & \end{array}$$

Conclusion: Stability of the system requires $0 < K < 8$

Example 2: To find out the acceptable range of $k > 0$ and $a > 0$ for the controlled system to be stable



Characteristic polynomial:

$$q(s) = s^4 + 6s^3 + 11s^2 + (k+6)s + k\,a$$

Routh-Hurwitz array:

$$\begin{array}{c|lll} 1 & 1 & 11 & k\,a \\ 2 & 6 & k+6 & \\ 3 & b_1 & k\,a & \\ 4 & c_1 & & \\ 5 & k\,a & & \end{array}$$

with

$$b_1 = \frac{60-k}{6}, \qquad c_1 = \frac{b_3(k+6) - 6k\,a}{b_3}$$

Conclusion: $b_1 > 0$ and $c_1 > 0 \Rightarrow k < 60$ and

$$a < \frac{(60-k)(k+6)}{36k}$$

Chapter 5. Steady-State Error

Objective: To know whether or not the response of a system can approach to the reference signal as time increases

Method: Transfer function analysis + Final-value theorem

Assumption: Systems are stable

5.1 Definition Steady-state error

$$e(t) = r(t) - y(t) \qquad \text{for} \qquad t \to \infty$$

where $r(t)$ – reference signal; $y(t)$ – output

Final value theorem:

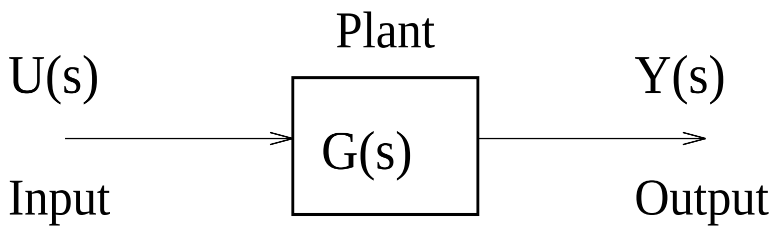
$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$E(s)$ – Laplace transform of $e(t)$

Remark: In many cases, by properly designing a controller, if a controlled system has $e(\infty) = 0$, then $e(t) \approx 0$ for $t > T_s$ is also ensured

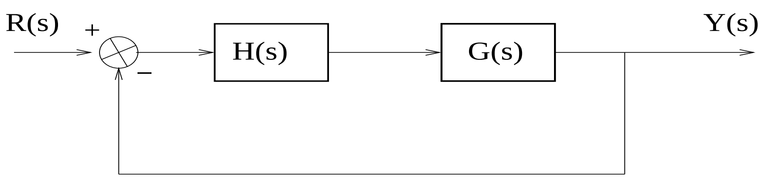
5.2 Feedback Control

Open-loop system

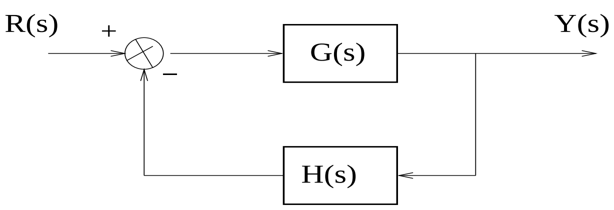


Disadvantages: Uncertainties & disturbances can affect system's performance heavily

Closed-loop system: To adjust input to the system according to the tracking error signal



(a)

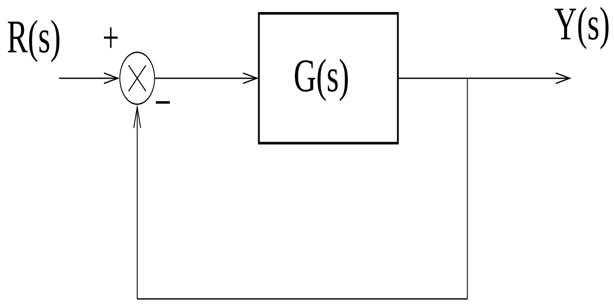


(b)

(a) A controller $H(s)$ in the forward channel

(b) A controller $H(s)$ in the backward channel

5.3 Unity Feedback Control Structure



Converting other two typical feedback structures (a) & (b) into the unity feedback structure:

- $\bar{G} = GH$ for the structure (a)
- $\bar{G} = \dfrac{G}{1 + G(H-1)}$ for the structure (b)

<u>Remarks</u>:

- For the structure (b), $\bar{G}$ is obtained from the relation
$$\frac{G}{1+GH} = \frac{\bar{G}}{1+\bar{G}}$$
- Only the unity feedback structure will be considered in the analysis of steady-state error

5.4 Steady-State Error Analysis

System types Rewrite as the transfer function

$$G(s) = \frac{1}{s^k} \cdot \frac{c_{\bar{m}} s^{\bar{m}} + \cdots + c_1 s + c_0}{d_{\bar{n}} s^{\bar{n}} + \cdots + d_1 s + d_0}$$

where c_0 & d_0 are non-zero

System *type* = the integer k = the number of the poles of value zero

Input types: Step, ramp & acceleration signals

Input types	$u(t)$	$U(s)$
Step	1	$1/s$
Ramp	t	$1/s^2$
Acceleration	$t^2/2$	$1/s^3$

Steady-State Error Formula

$$E(s) = R(s) - Y(s) = R(s) - G(s)E(s)$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s)}$$

Applying the final-value theorem to $E(s) \Rightarrow$

$$e(\infty) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

Apply the formula to the three input cases:

Case 1: Step Input

$$e(\infty) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+G(0)}$$

The value of $G(0)$ depends on types of $G(s)$:

$$G(0) = \begin{cases} \frac{c_0}{d_0}, & k=0 \\ \infty, & k>0 \end{cases}$$

$\Rightarrow$, for the step input,

$$e(\infty) = \begin{cases} \frac{d_0}{c_0+d_0}, & k=0 \quad \text{(Type 0 systems)} \\ 0, & k>0 \quad \text{(Systems of type 1,2, ...)} \end{cases}$$

Case 2: Ramp Input: For $R(s) = 1/s^2$,

$$\begin{aligned} e(\infty) &= \lim_{s \to 0} \frac{sR(s)}{1+G(s)} \\ &= \lim_{s \to 0} \frac{1}{s + sG(s)} = \lim_{s \to 0} \frac{1}{sG(s)} \end{aligned}$$

As

$$\lim_{s \to 0} sG(s) = \begin{cases} 0, & k = 0 \\ \frac{c_0}{d_0}, & k = 1 \\ \infty, & k > 1 \end{cases}$$

$\Rightarrow$

$$e(\infty) = \begin{cases} \infty, & k = 0 \\ \frac{d_0}{c_0}, & k = 1 \\ 0, & k > 1 \end{cases}$$

Case 3: Acceleration Input: For $R(s) = 1/s^3$,

$$\begin{aligned} e(\infty) &= \lim_{s \to 0} \frac{sR(s)}{1+G(s)} \\ &= \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)} \\ &= \lim_{s \to 0} \frac{1}{s^2 G(s)} \end{aligned}$$

As

$$\lim_{s\to 0} s^2 G(s) = \begin{cases} 0, & k = 0, 1 \\ \frac{c_0}{d_0}, & k = 2 \\ \infty, & k > 2 \end{cases}$$

$\Rightarrow$

$$e(\infty) = \begin{cases} \infty, & k = 0, 1 \\ \frac{d_0}{c_0}, & k = 2 \\ 0, & k > 2 \end{cases}$$

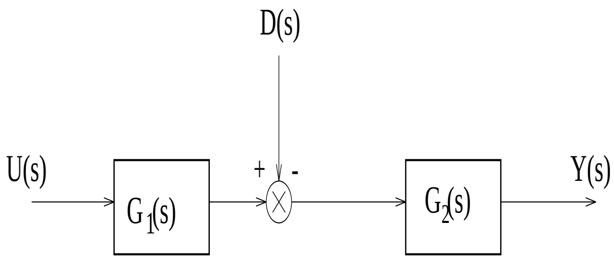
Summary of Steady-State Error:

System type	Step input	Ramp input	Acc. input
0	$\frac{d_0}{c_0+d_0}$	∞	∞
1	0	$\frac{d_0}{c_0}$	∞
2	0	0	$\frac{d_0}{c_0}$

5.5 Disturbance Resistance

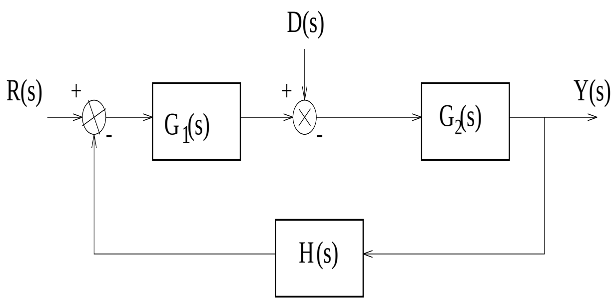
Advantage of feedback control: Disturbance resistance

Open-loop system:



$$Y(s) = G_2(s)G_1(s)R(s) - G_2(s)D(s)$$

Closed-loop system:



$$Y(s) = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}D(s)$$

DC motor control system

$$
\begin{aligned}
G_1(s) &= \frac{K_r K K_m}{R_f}, \\
G_2(s) &= \frac{1}{s(Js + f)}, \\
H(s) &= K_f / K_r
\end{aligned}
$$

Conclusion

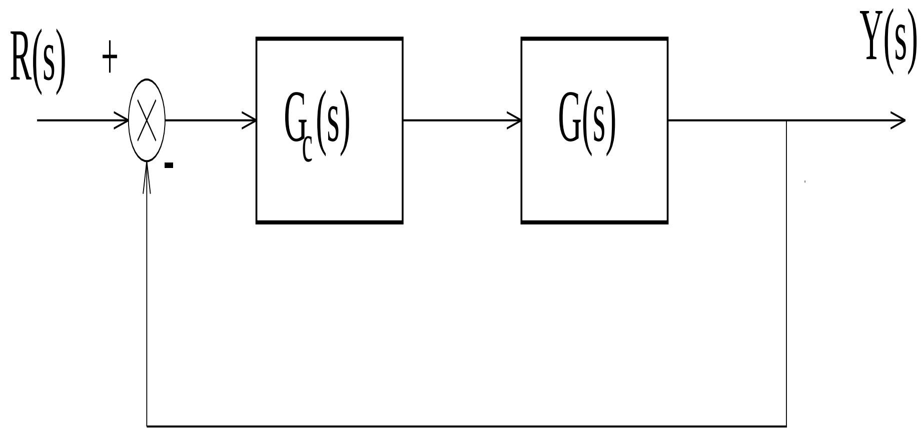
- Open-loop system ($H(s) = 0$) cannot work in the presence of load changes since $G_2(0) = \infty$
- Closed-loop system has disturbance-to-output gain $\frac{G_2(0)}{1+G_1(0)G_2(0)H(0)} = \frac{R_f}{K_r K K_m K_f}$

Chapter 6. PID Control

Objective: Introduction of PID controller & its experiemntal design methods

- P – proportional
- I – integral
- D – derivative

PID controller in the loop



- Input to PID controller – error signal

$$e(t) = r(t) - y(t)$$

- Output of PID controller – input to the plant

Time-domain description:
Integro-differential equation

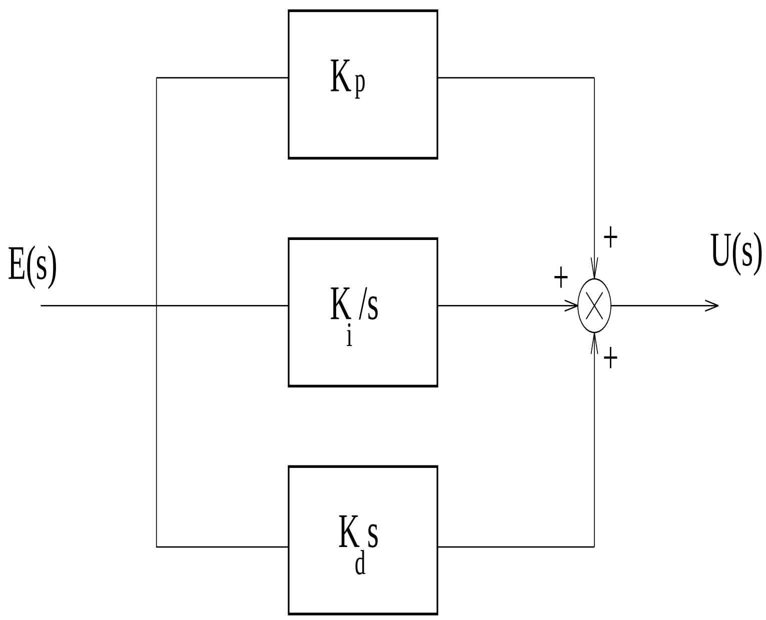
$$u(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{de(t)}{dt}$$

K_p, K_i & K_d – P, I & D parameters

s-domain description: Transfer function

$$\frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s$$

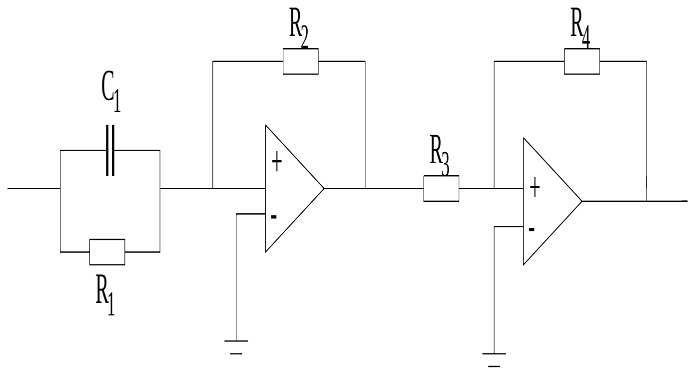
Block diagram



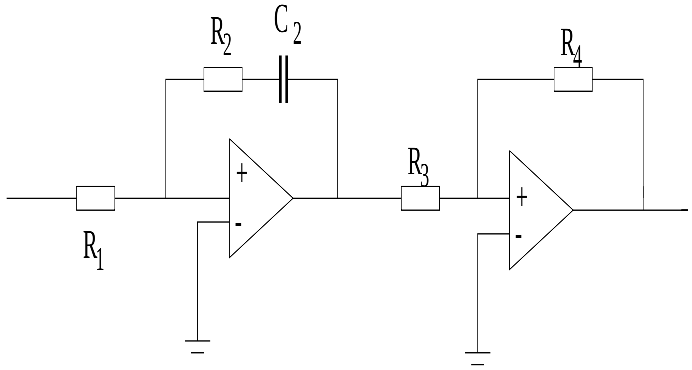
6.1. Implementation of PID Controller

Physical implementation of PID control:
Electronic, mechanical, hydraulic, digital (on computers), etc.

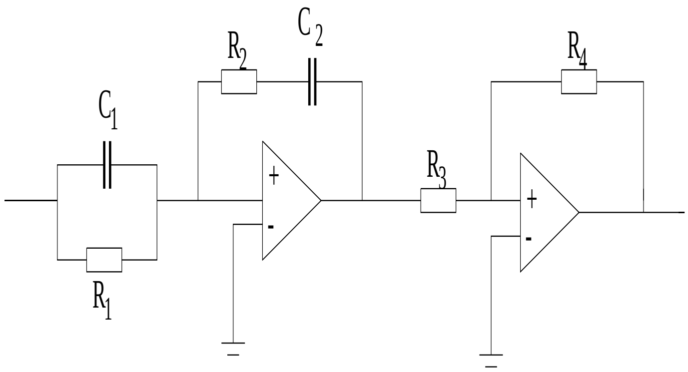
Operational amplifier circuit implementation



PD controller with $G_c(s) = \dfrac{R_4 R_2}{R_3 R_1}(R_1 C_1 s + 1)$



PI controller with $G_c(s) = \dfrac{R_4(R_2 C_2 s + 1)}{R_3 R_1 C_2 s}$



PID controller with

$$G_c(s) = \frac{R_4(R_1C_1s+1)(R_2C_2s+1)}{R_3R_1C_2s}$$

<u>Remark</u> Derivation of transfer functions for the OPs' circuits is quite simple under assumptions:

- Currents through OPs and
- Input voltages to OPs

are considerable smaller than other currents and voltages in circuits.

6.2. Effect of PID Control

<u>P control</u>: To reduce the error $e(t) = r(t) - y(t)$: Small error $\Rightarrow$ large control signal $u(t)$

<u>I control</u>: Towards zero error: $e(t) \to 0 \Rightarrow u(t)$ may be large

<u>D control</u>: Predict change of error: When $e(t) \to 0$, but $\dot{e}(t) \neq 0 \Rightarrow u(t)$ may be large

<u>Remarks</u>: It is not suitable to use pure D or I control for the reasons:

- D control: Sensitive to noices/disturbances & reduce system type by one
- P control: Slow down response

<u>Practical use</u>: P control or a combined PI, PD or PID control

6.3. Design of PID Controller

Objective: Determination of PID parameters

Trial-Error Method: An experimental method

Rules of thumb:

$$K_p > K_i > K_d,$$
$$K_p \approx (5 \sim 10)K_i,$$
$$K_i \approx (5 \sim 10)K_d$$

Procedure:

Step 1: Set $K_i = 0$ & $K_d = 0$. Increase K_p from zero;

Step 2: Fix K_p. Increase K_i from zero;

Step 3: Fix K_p & K_i. Increase K_d from zero.

Note: Several iterations of the procedure may be necessary

Remarks on trial-&-error method:

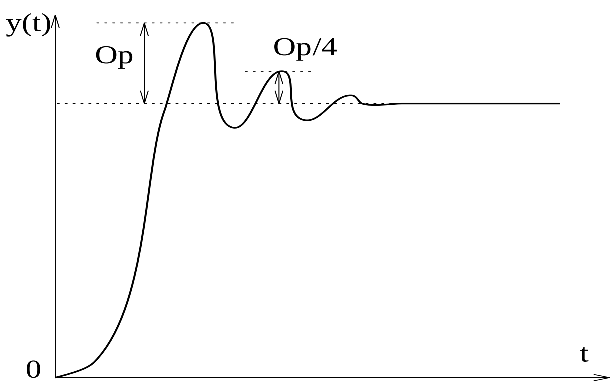
- Adavantages: Simple
- Disadvantages:
  - unsatisfactory performance
  - expensive on-site experiment
  - issues of equipment safety

Ziegler-Nichols Method:

An empirical method involving limited experiment

Basic idea: PID parameter settings are much the same if a common optimum performance index had been used

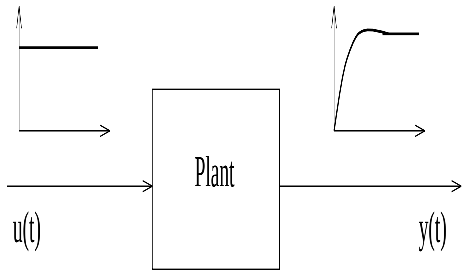
Common performance: *Quarter-decay* response



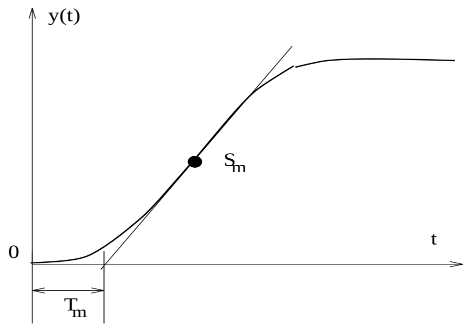
1st Ziegler-Nichols Method

Restriction: For systems with no overshoot of step response

Producing step response of open-loop system:



Determination of two parameters:



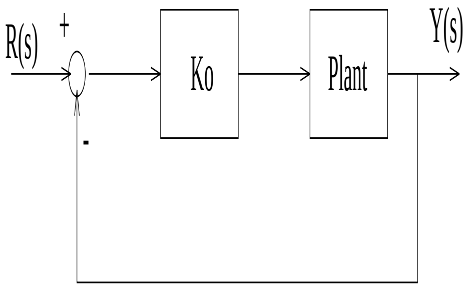
PID parameter settings:

Control	K_p	K_i	K_d
P	$\frac{1}{S_m T_m}$	0	0
PI	$\frac{0.9}{S_m T_m}$	$\frac{3}{S_m}$	0
PID	$\frac{1.2}{S_m T_m}$	$\frac{2.4}{S_m}$	$\frac{0.6}{S_m}$

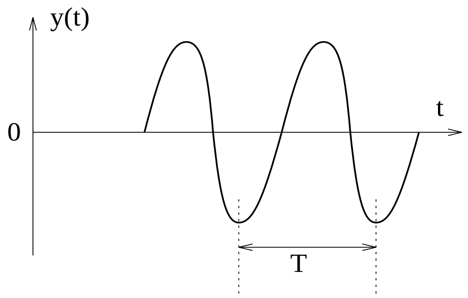
2nd Ziegler-Nichols Method

Restriction: For systems of order > 2

Producing oscillation using P control:



Determination of period of oscillation:



PID parameter settings:

Control	K_p	K_i	K_d
P	$\frac{K_o}{2}$	0	0
PI	$\frac{9K_o}{20}$	$\frac{3K_oT_o}{8}$	0
PID	$\frac{3K_o}{5}$	$\frac{3K_oT_o}{10}$	$\frac{3K_oT_o}{40}$

Restriction on Applying Ziegler-Nichols Methods

1st method applies if the step response of the open-loop system has no overshoot

2nd method implies the underlying system should be of at least third order

Example: Analytical determination of K_0 & T_0

3rd-order system:

$$G(s) = \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Characteristic equation of the closed-loop system:

$$s^3 + a_2 s^2 + a_1 s + a_0 + b_0 K_0 = 0$$

Setting $s = \pm \omega_0$: $\Rightarrow$

$$\mp j\omega_0^3 - a_2\omega_0^2 \pm a_1 j\omega_0 + b_0 K_0 = 0$$

which has solution

$$\omega_0 = \sqrt{a_1}\,, \qquad K_0 = \frac{a_2 a_1 - a_0}{b_0}$$