MM5 Stability Analysis

Readings:

- Section 4.4 (stability, p.212-223);
- Section 4.3 (steady-state tracking & system type, p.200-210)
- Section 3.5 (effects of zeros & add. Poles, p.131-138)
- Extra reading materials (p.40-60)

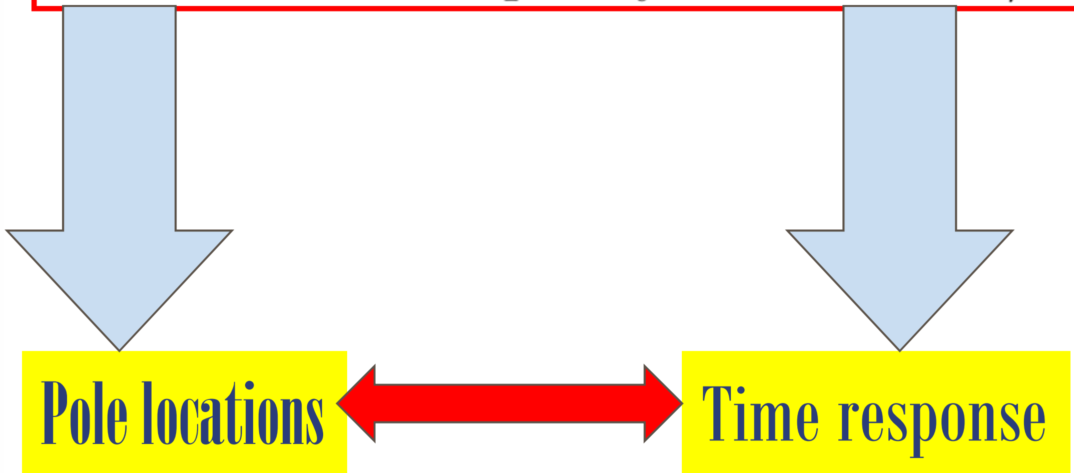
What have we talked in **MM4**?

- Poles vs time reponses
- Feedback charactersitics
- Matlab: pzmap(), sgrid

MM4 : Poles vs Performance

$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

- ξ – *damping ratio*, a dimensionless factor
- ω_n – *natural frequency* with unit rad/s



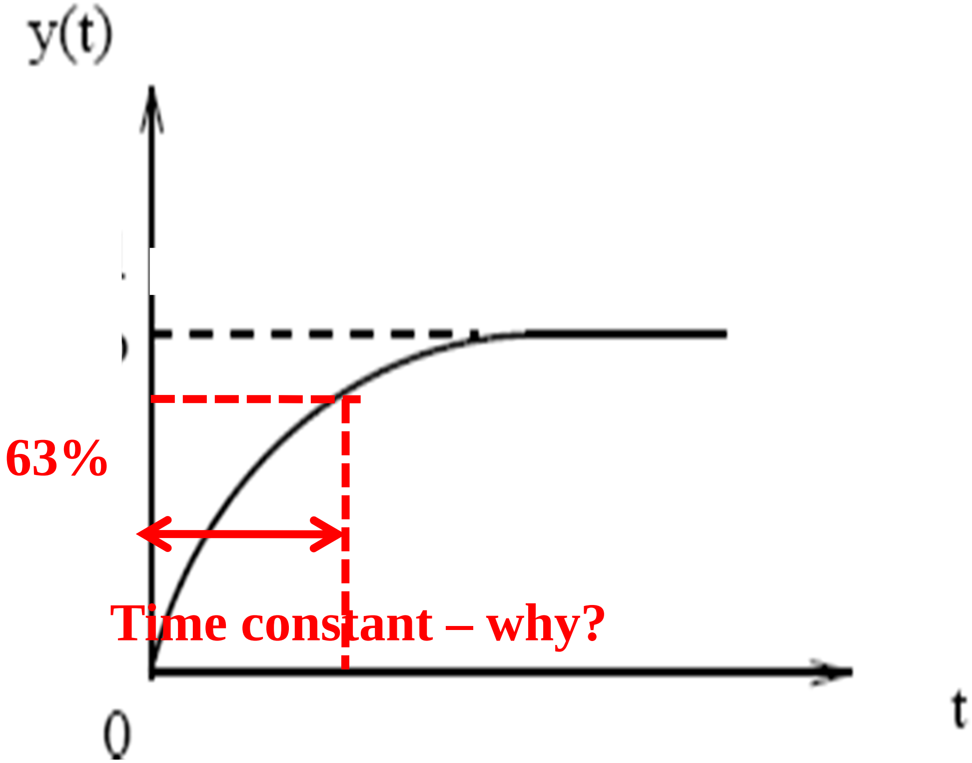
MM4: First-order System

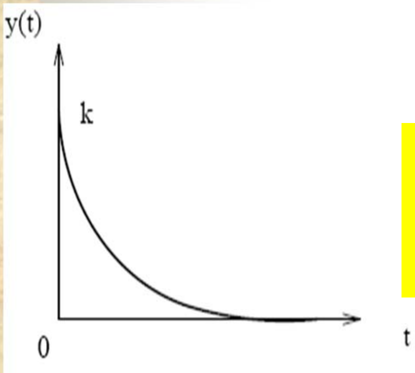
$$G(s)=\frac{1}{\tau s+1}, \qquad \text{assume} \quad \tau>0$$

pole: $-\frac{1}{\tau}$, time constant: τ,

Impulse response: $y(t)=L(\frac{1}{\tau s+1})=e^{-\frac{1}{\tau}t}$

Step response: $y(t)=L(\frac{1}{s(\tau s+1)})=1-e^{-\frac{1}{\tau}t}$





Time response is determined by the time constant

System pole is the negative of inverse time constant

MM4: Second-Order System

$$G(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2},\quad \text{assume } \omega_n>0,\quad \xi\ge 0$$

poles: $p_{1,2}=-\xi\omega_n\pm\omega_n\sqrt{\xi^2-1}$

real (different) poles: $p_{1,2}=-\xi\omega_n\pm\omega_n\sqrt{\xi^2-1}$, if $\xi>1$

real (identical) poles: $p_{1,2}=-\xi\omega_n$, if $\xi=1$

complex poles: $p_{1,2}=-\xi\omega_n\pm j\omega_n\sqrt{1-\xi^2}$, if $0<\xi<1$

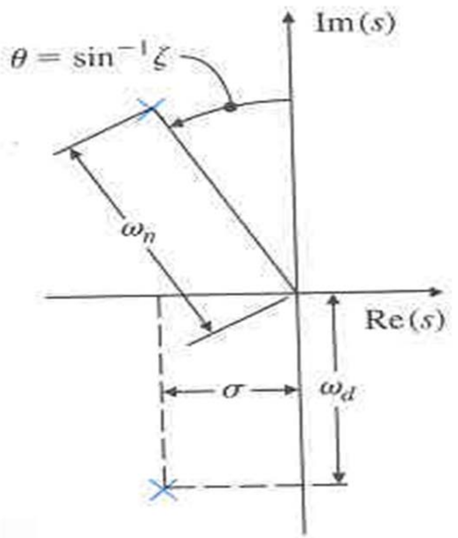
complex poles: $p_{1,2}=\pm j\omega_n$, if $\xi=0$

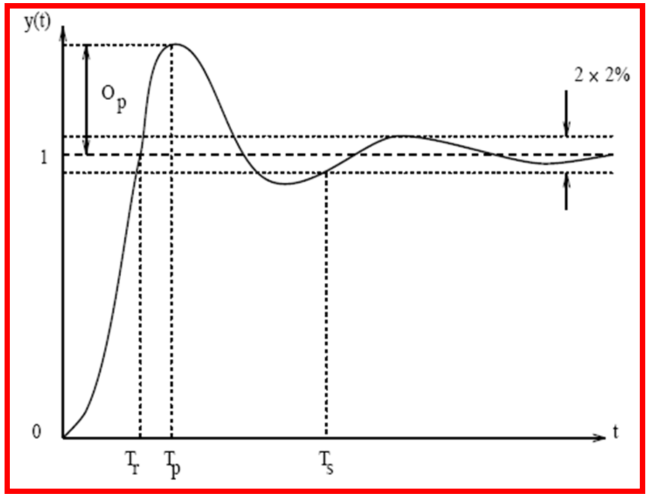
$$t_r\cong\frac{1.8}{\omega_n}$$

$$t_s\cong\frac{4.6}{\zeta\omega_n}\cong\frac{4.6}{\sigma}$$

$$M_p\cong\begin{cases}5\%, & \zeta=0.7\\ 16\%, & \zeta=0.5\\ 35\%, & \zeta=0.3\end{cases}$$

$$t_p\cong\frac{\pi}{\omega_d},\quad \omega_d=\omega_n\sqrt{1-\zeta^2}$$





Rise time $T_r=\frac{\pi-\phi}{\omega_d}$

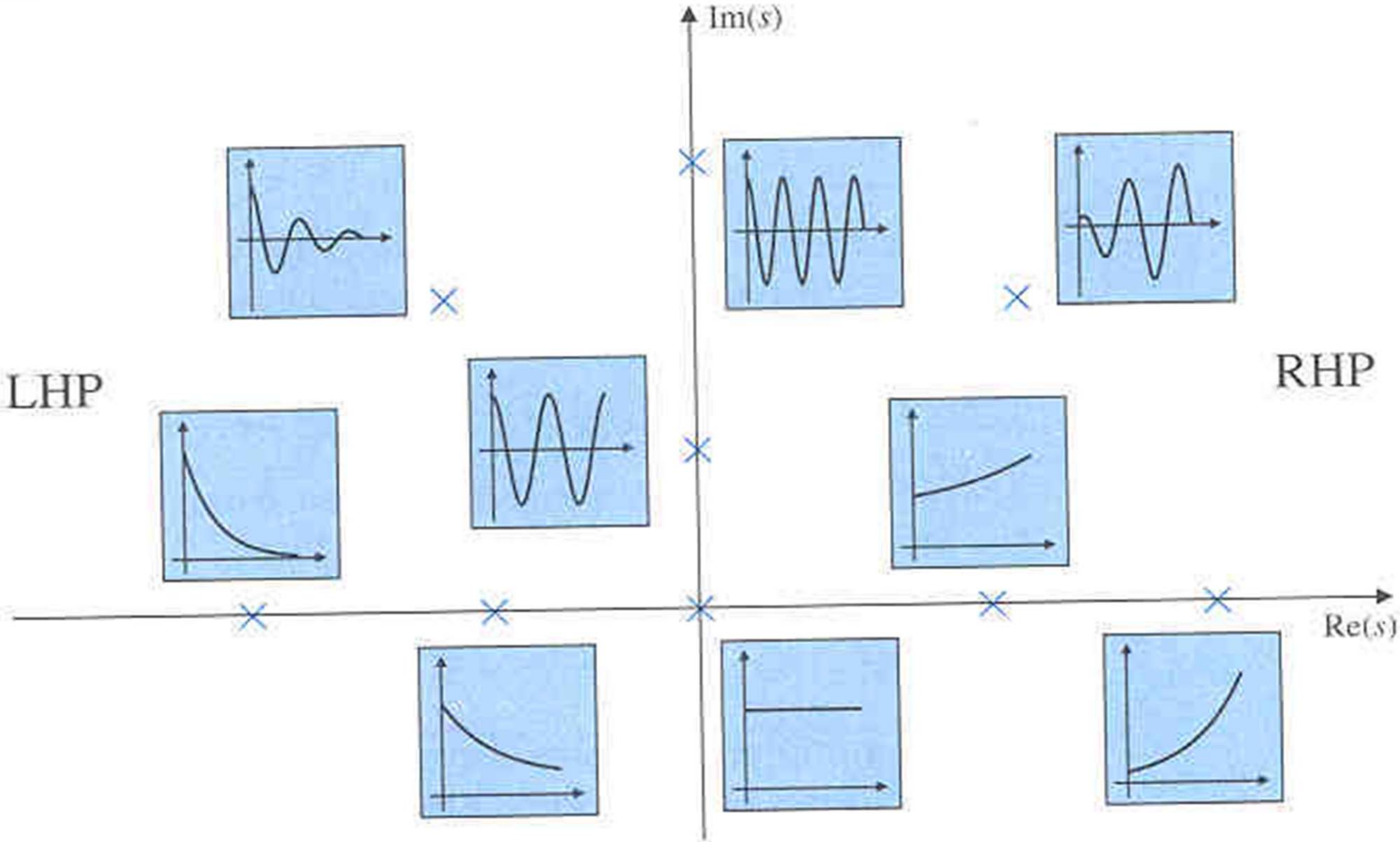
Peak time $T_p=\frac{\pi}{\omega_d}$

Settling time $T_s\approx\frac{4}{\xi\omega_n}$

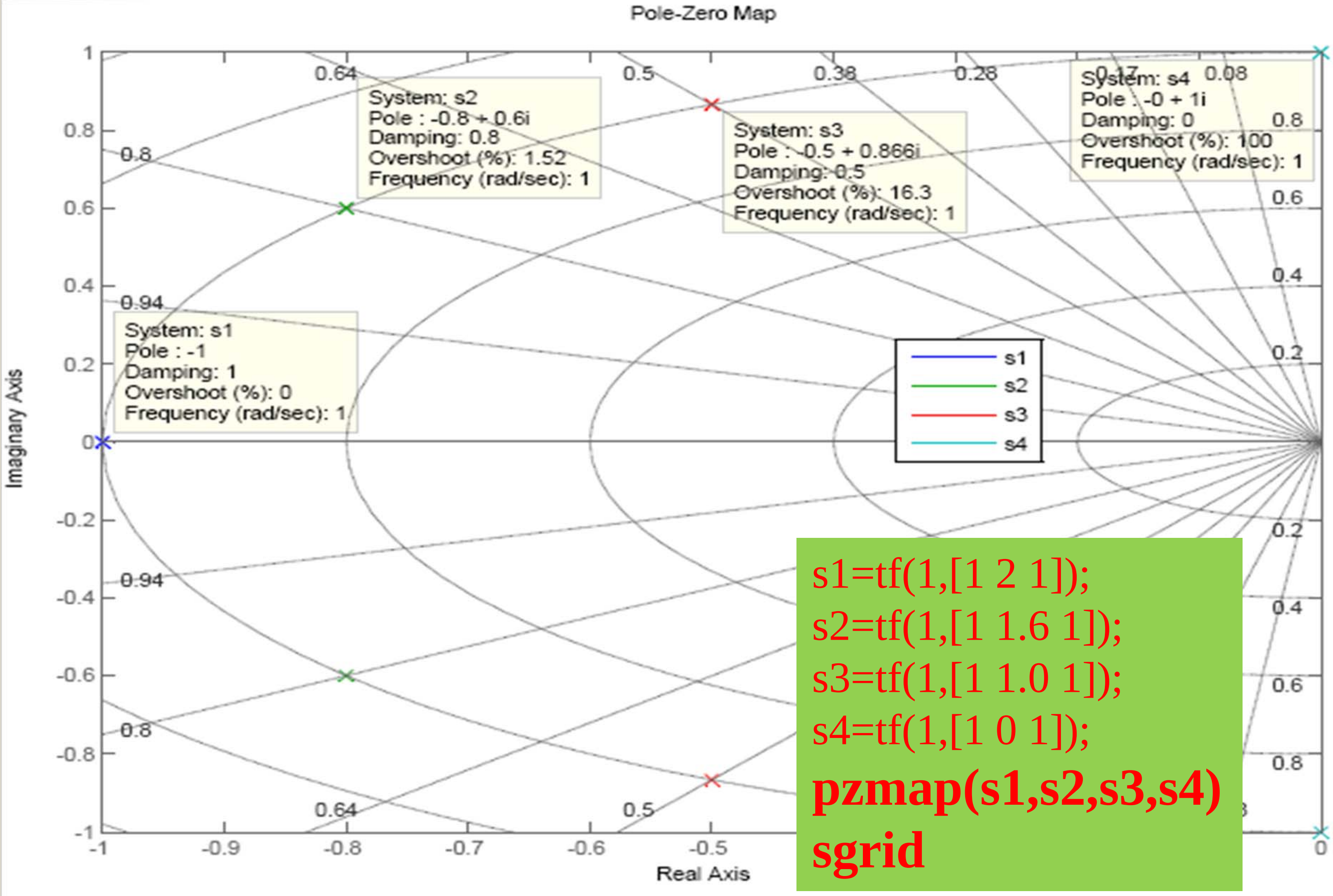
Overshoot $O_p=e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}$

$\phi=\tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$, $\omega_d=\omega\sqrt{1-\xi^2}$

MM4: Summary of Pole vs Performance



MM4: Plot of Pole Locations




```
s1=tf(1,[1 2 1]);
s2=tf(1,[1 1.6 1]);
s3=tf(1,[1 1.0 1]);
s4=tf(1,[1 0 1]);
pzmap(s1,s2,s3,s4)
sgrid
```

Goals for this lecture (MM5)

- **Stability analysis**
  - **Definition of BIBO**
  - **Pole locations**
  - **Routh criteron**
- Steady-state errors
  - Final Theorem
  - DC-Gain
  - Stead-state errors
- Effects of zeros and additional poles

MM4: Summary of Pole vs Performance

$$G(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2}, \quad \text{assume } \omega_n>0, \quad \xi\ge 0$$

poles: $p_{1,2}=-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

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real (identical) poles: $p_{1,2}=-\xi\omega_n$, if $\xi=1$

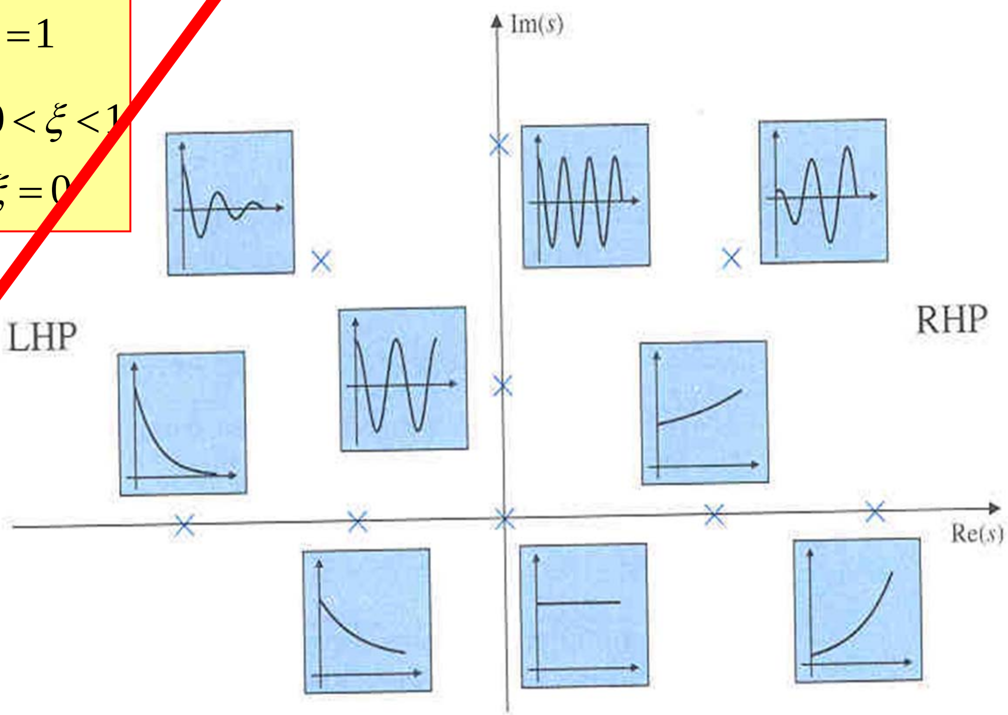
complex poles: $p_{1,2}=-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$, if $0<\xi<1$

complex poles: $p_{1,2}=\pm j\omega_n$, if $\xi=0$

How about if these are not satisfied?

$$G(s)=\frac{1}{\tau s+1}, \quad \text{assume } \tau>0$$

pole: $-\frac{1}{\tau}$, time constant: τ,



System Stability

- Definitions
  - **BIBO stability**
  - Internal stability
  - …

- Determination methods:

  - ❑ **Impulse response function/sequence**
  - ❑ **Roots of characteristic equation (poles)**
  - ❑ **Routh's stability criterion**
  - ❑ Gain and phase margins
  - ❑ Nyquist stability criterion

BIBO Stability

- A system is said to have **bounded input-bounded output (BIBO) stability** if every bounded input results in a bounded output (regardless of what goes on inside the system)

- The continuous (LTI) system with impuse response **h(t)** is BIBO stable if and only if **h(t)** is absolutely integrallable

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

$$L(h(t)) = H(s)$$

- All system poles locate in the left half s-plane

BIBO Stability – **Characteristic Equation**

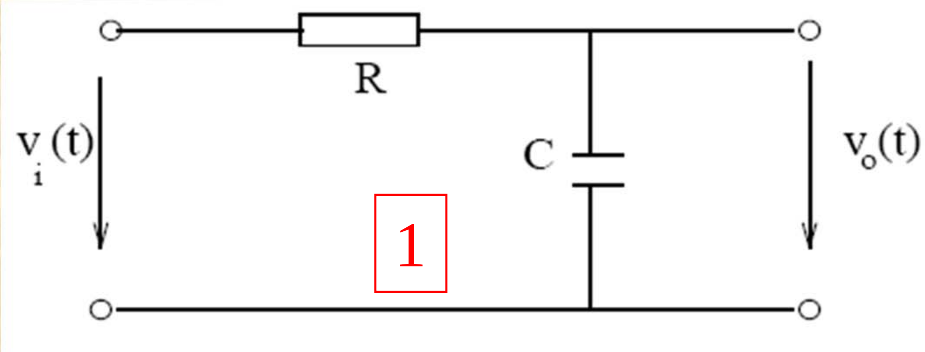
- **Characteristic equation**

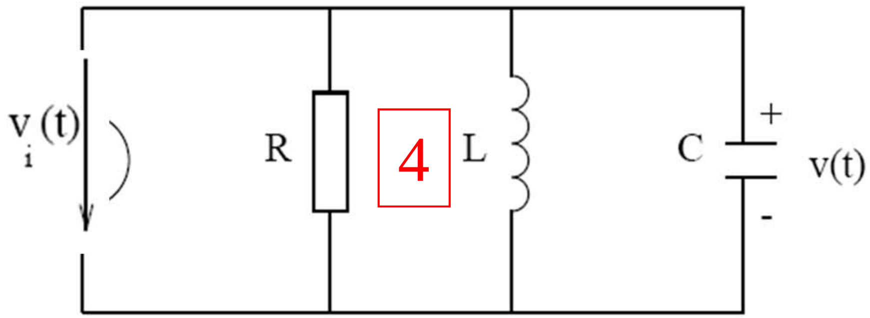
$$G(s) = \frac{\sum_{i=0}^{m} b_i s^i}{\sum_{i=0}^{n} a_i s^i}, \quad \text{characteri stic eauqtion} : \sum_{i=0}^{n} a_i s^i = 0$$

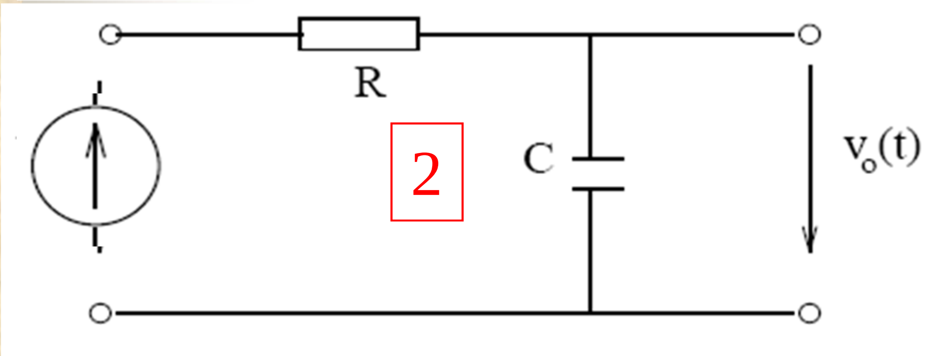
$$G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n + \omega_n^2}, \quad \text{characteri stic eauqtion} : s^2 + 2\xi\omega_n + \omega_n^2 = 0$$

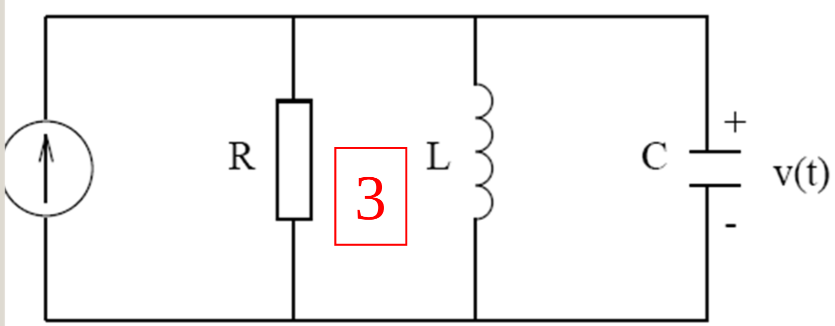
- All poles (roots of the chracteristic equation) of the continuous system are strictly in the LHP of the s-plane - **asymptotic internal stability**
- **(Matlab: roots(den))**

BIBO Stability – **Execise (I)**









- **Are these systems BIBO stable?**
- Intuitive explanation
- Theoretical analysis

BIBO Stability – **Routh Criterion (I)**

- **Motivation:** Testing stability without calculating poles
- **Criterion:** For a stable system, there is no changes in sign and no zeros in the first column of the Routh array.

Characteristic polynomial $q(s)$:

$$a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh-Hurwitz Array

$$\begin{array}{l|llllll}
1 & a_n & a_{n-2} & a_{n-4} & \cdots & \cdots & a_0 \\
2 & a_{n-1} & a_{n-3} & a_{n-5} & \cdots & a_1 & \\
3 & b_1 & b_2 & \cdots & \cdots & b_{n/2} & \\
4 & c_1 & c_2 & \cdots & c_{n/2-1} & & \\
\vdots & \vdots & \vdots & & & & \\
n & g_1 & g_2 & & & & \\
n+1 & h_1 & & & & &
\end{array}$$

$$\begin{array}{cccc}
\times & \times & \times & \times \\
\times & a & b & \times \\
\times & c & d & \times \\
\times & e & \times & \times
\end{array} \Rightarrow \quad e = -\frac{1}{c}\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{bc - ad}{c}$$

BIBO Stability – **Routh Criterion (II)**

Routh-Hurwitz criterion: No. of unstable roots of $q(s)$ = No. of changes in sign of the 1st column

- One/more zeros appearing in the 1st column $\Rightarrow$ poles with zero real part
- Marginally stable if the poles with zero real part are distinct
- Unstable if these poles are repeated

BIBO Stability – **Examples**

2nd-order system

$$q(s) = a_2 s^2 + a_1 s + a_0$$

Routh-Hurwitz array

$$\begin{array}{c|cc} 1 & a_2 & a_0 \\ 2 & a_1 & 0 \\ 3 & b_1 & 0 \end{array} \quad \Rightarrow \quad \begin{array}{c|cc} 1 & a_2 & a_0 \\ 2 & a_1 & 0 \\ 3 & a_0 & 0 \end{array}$$

as $b_1 = -\frac{1}{a_1}\begin{vmatrix} a_2 & a_0 \\ a_1 & 0 \end{vmatrix} = \frac{a_1 a_0 - a_2 \cdot 0}{a_1} = a_0$

Conclusion: 2nd-order system is stable $\Leftrightarrow a_2,\ a_1,\ a_0$ have the same sign

3rd-order system

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 .$$

Routh-Hurwitz array:

$$\begin{array}{c|cc} 1 & a_3 & a_1 \\ 2 & a_2 & a_0 \\ 3 & b_1 & 0 \\ 4 & c_1 & 0 \end{array} \quad \Rightarrow \quad \begin{array}{c|cc} 1 & a_3 & a_1 \\ 2 & a_2 & a_0 \\ 3 & \frac{a_2 a_1 - a_3 a_0}{a_2} & 0 \\ 4 & a_0 & 0 \end{array}$$

Conclusion: Stability of 3rd-order system $\Leftrightarrow$

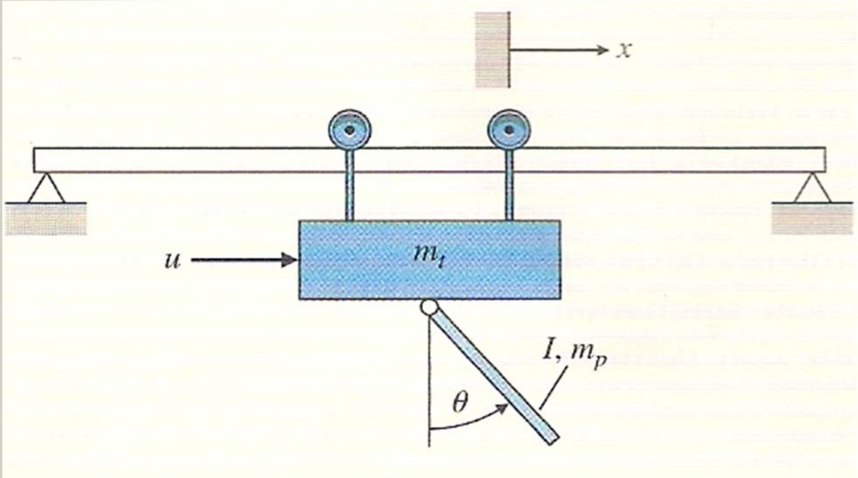
(1) $a_3,\ a_2,\ a_1,\ a_0$ have the same sign;

(2) $a_2 a_1 > a_3 a_0$.

See page 46-49 on the extra readings

BIBO Stability – **Execise (II)**

- How about the stability of your project systems?

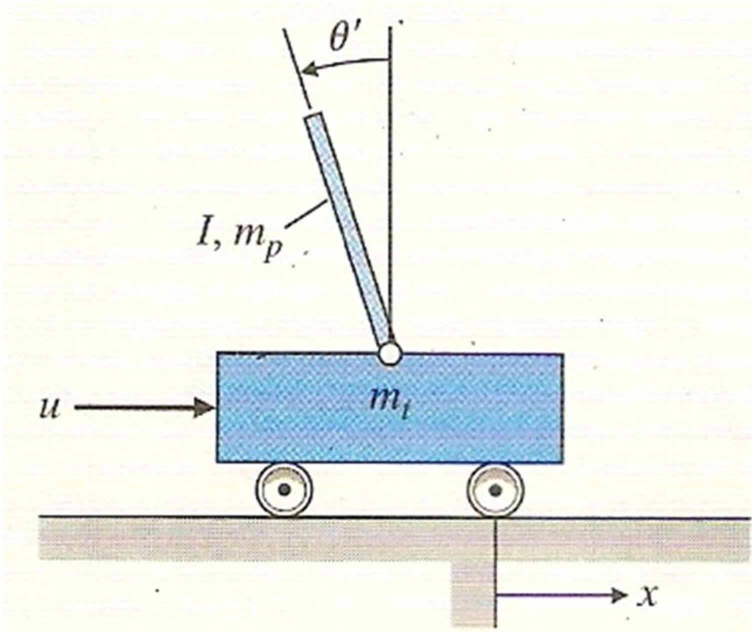


$$(I + m_p l^2)\ddot{\theta}' - m_p g l \theta' = m_p l \ddot{x}$$

$$(m_t + m_p)\ddot{x} + b\dot{x} - m_p l \ddot{\theta}' = u.$$

$$(I + m_p l^2)\ddot{\theta} + m_p g l \theta = -m_p l \ddot{x}$$

$$(m_t + m_p)\ddot{x} + b\dot{x} + m_p l \ddot{\theta} = u.$$



BIBO Stability – **Objectives of Control**

Control design Objectives:

- Closed-loop stability
- Good command response
- Disturbance attenuation
- Robustness

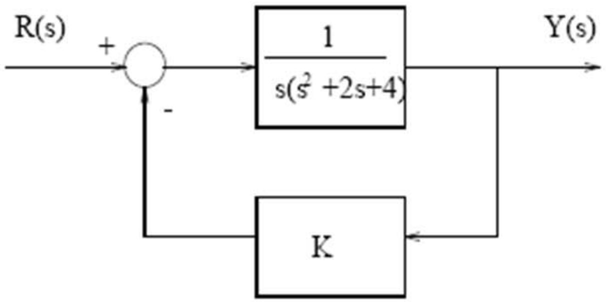
Performance:

- **time domain specifications**
- **Frequency specifications**
- **Dynamic transient responses**
- **Steady-state responses**
- **Continuous control systems**
- **Digital control systems**

See page 49-50 on the extra readings

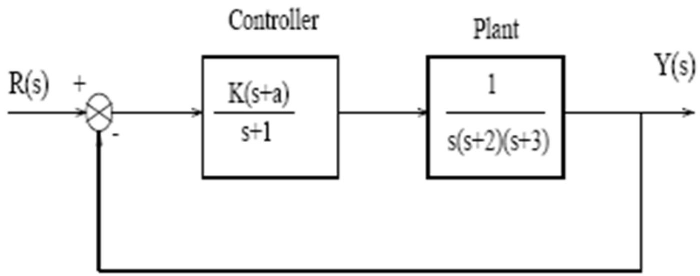
BIBO Stability – **Stabilizing Control**

Example 1: Determine the controller gain K to stablise the 3rd-order system



Conclusion: Stability of the system requires $0 < K < 8$

Example 2: To find out the acceptable range of $k > 0$ and $a > 0$ for the controlled system to be stable



Conclusion: $b_1 > 0$ and $c_1 > 0 \Rightarrow k < 60$ and

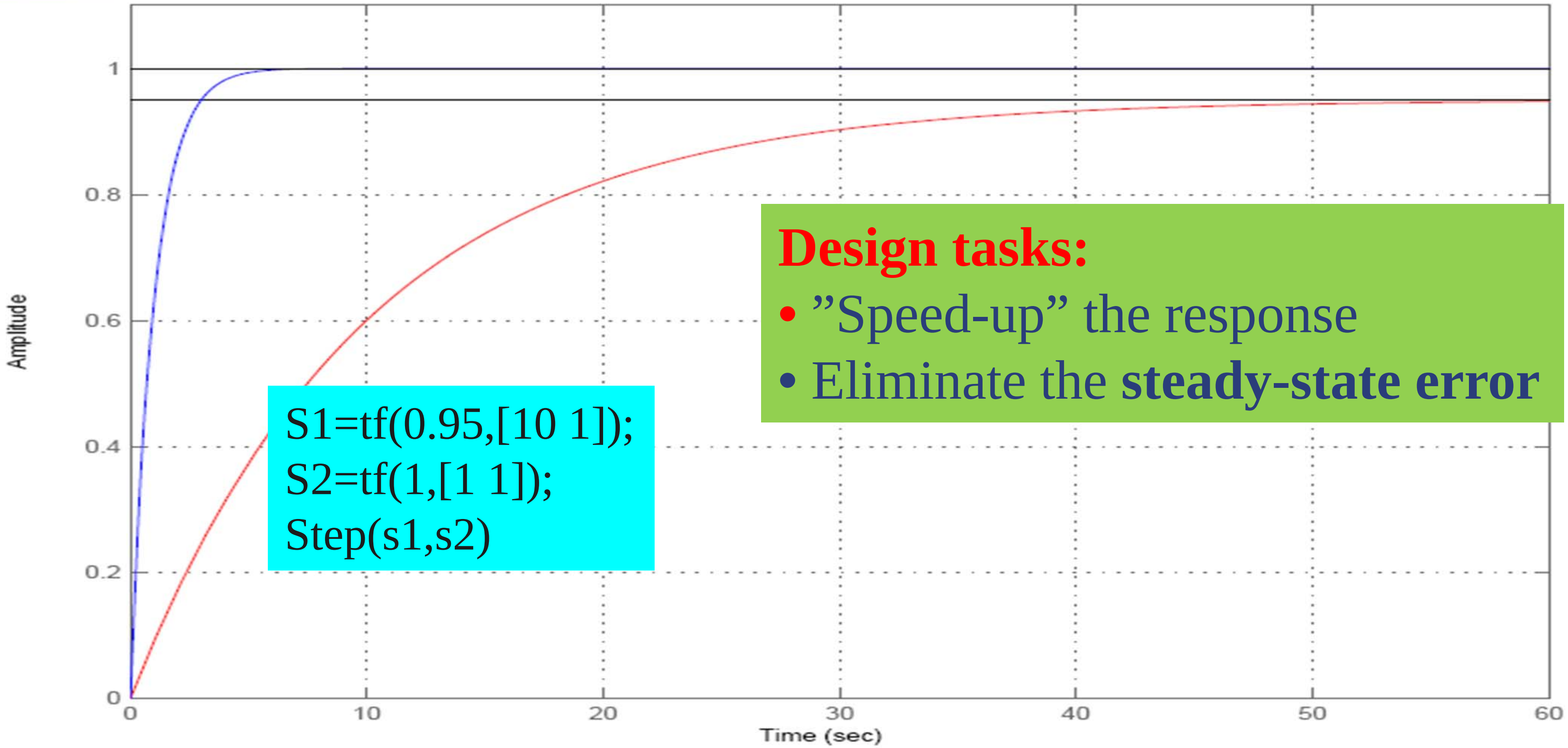
$$a < \frac{(60-k)(k+6)}{36k}$$

Goals for this lecture (MM5)

- Stability analysis
  - Definition of BIBO
  - Pole locations
  - Routh criteron
- **Steady-state errors**
  - **Final Theorem**
  - **DC-Gain**
  - **Stead-state errors**
- Effects of zeros and additional poles

MM4: Example : First-order System

- An design problem: control sys1 so as to have the same performance as sys2




```
S1=tf(0.95,[10 1]);
S2=tf(1,[1 1]);
Step(s1,s2)
```

Design tasks:

- ”Speed-up” the response
- Eliminate the **steady-state error**

Steady-State Error

- **Objective:**

  to know whether or not the response of a system can approach to the reference signal as time increases

- **Assumption:**

  The considered system is stable

- **Analysis method:**

  Transfer function + final-value Theorem

Steady-State Error – **Final-Value Theorem**

Definition Steady-state error

$$e(t) = r(t) - y(t) \qquad \text{for} \qquad t \to \infty$$

where $r(t)$ – reference signal; $y(t)$ – output

<u>Final value theorem</u>:

$$e(\infty) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$E(s)$ – Laplace transform of $e(t)$

DC-Gain

$$e(\infty) = \lim_{s\to 0} s(R(s) - Y(s)) = \lim_{s\to 0} s(R(s) - G(s)R(s))$$

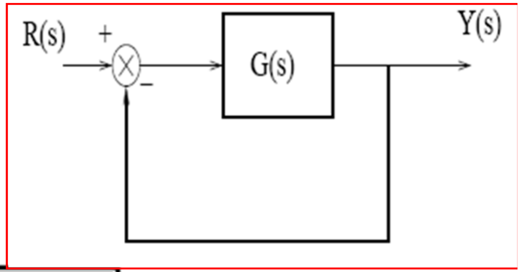
$$= \lim_{s\to 0} s(1 - G(s))R(s), \qquad R(s) = \frac{1}{s}$$

$$= \lim_{s\to 0} (1 - G(s)) = 1 - G(0)$$

```
S1=tf(0.95,[10 1]);
S2=tf(1,[1 1]);
Step(s1,s2)
```

Steady-State Error – **System Types**

- **Position-error constant** $K_p = \lim_{s \to 0} G_o(s)$
- **Velocity constant** $K_v = \lim_{s \to 0} sG_o(s)$
- **Acceleration constant** $K_a = \lim_{s \to 0} s^2 G_o(s)$
- System types (**type 0, type I, type II**)

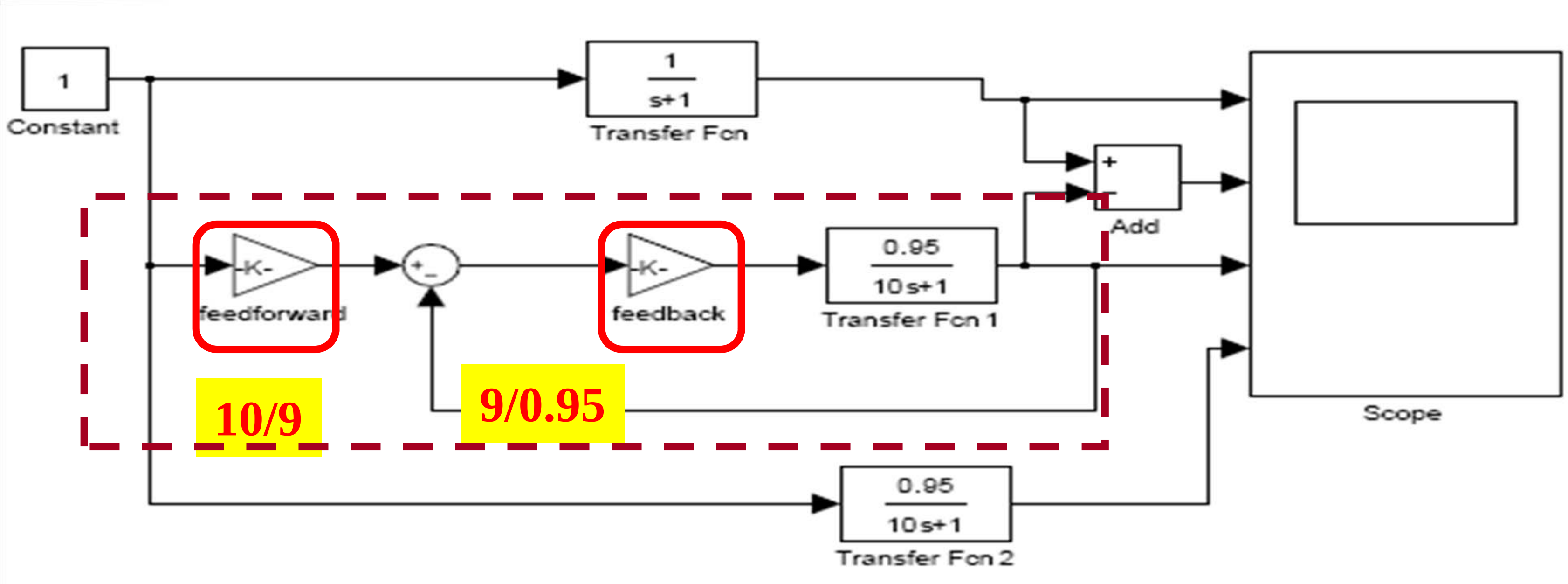


Type 0 systems	*Step Input*	*Ramp Input*	*Parabolic Input*
Steady State Error Formula	1/(1+Kp)	1/Kv	1/Ka
Static Error Constant	Kp = constant	Kv = 0	Ka = 0
Error	1/(1+Kp)	infinity	infinity

Type 1 systems	*Step Input*	*Ramp Input*	*Parabolic Input*
Static Error Constant	Kp = infinity	Kv = constant	Ka = 0
Error	0	1/Kv	infinity

Type 2 systems	*Step Input*	*Ramp Input*	*Parabolic Input*
Static Error Constant	Kp = infinity	Kv = infinity	Ka = constant
Error	0	0	1/Ka

Revisit of example: First-order System (II)



- What's the tpye of original system?
- Derive the transfer function of the closed-loop system
- What's the time constant and DC-gain of the CL system?
- What's the feedforward gain so that there is no steady-state error?

Goals for this lecture (MM5)

- Stability analysis
  - Definition of BIBO
  - Pole locations
  - Routh criteron
- Steady-state errors
  - Final Theorem
  - DC-Gain
  - Stead-state errors
- **Effects of zeros and additional poles**

System Zeros

- The dynamic behavior of a transfer function model can be characterized by the numerical value of its poles and zeros

$$G(s) = \frac{b_m(s-z_1)(s-z_2)\ldots(s-z_m)}{a_n(s-p_1)(s-p_2)\ldots(s-p_n)} \qquad (6\text{-}7)$$

- $\{z_i\}$ are the "**zeros**" and $\{p_i\}$ are the "**poles**"
- $n \geq m$ in order to have a physically realizable system

$G(s) = \dfrac{1}{\tau s+1}$, assume $\tau > 0$

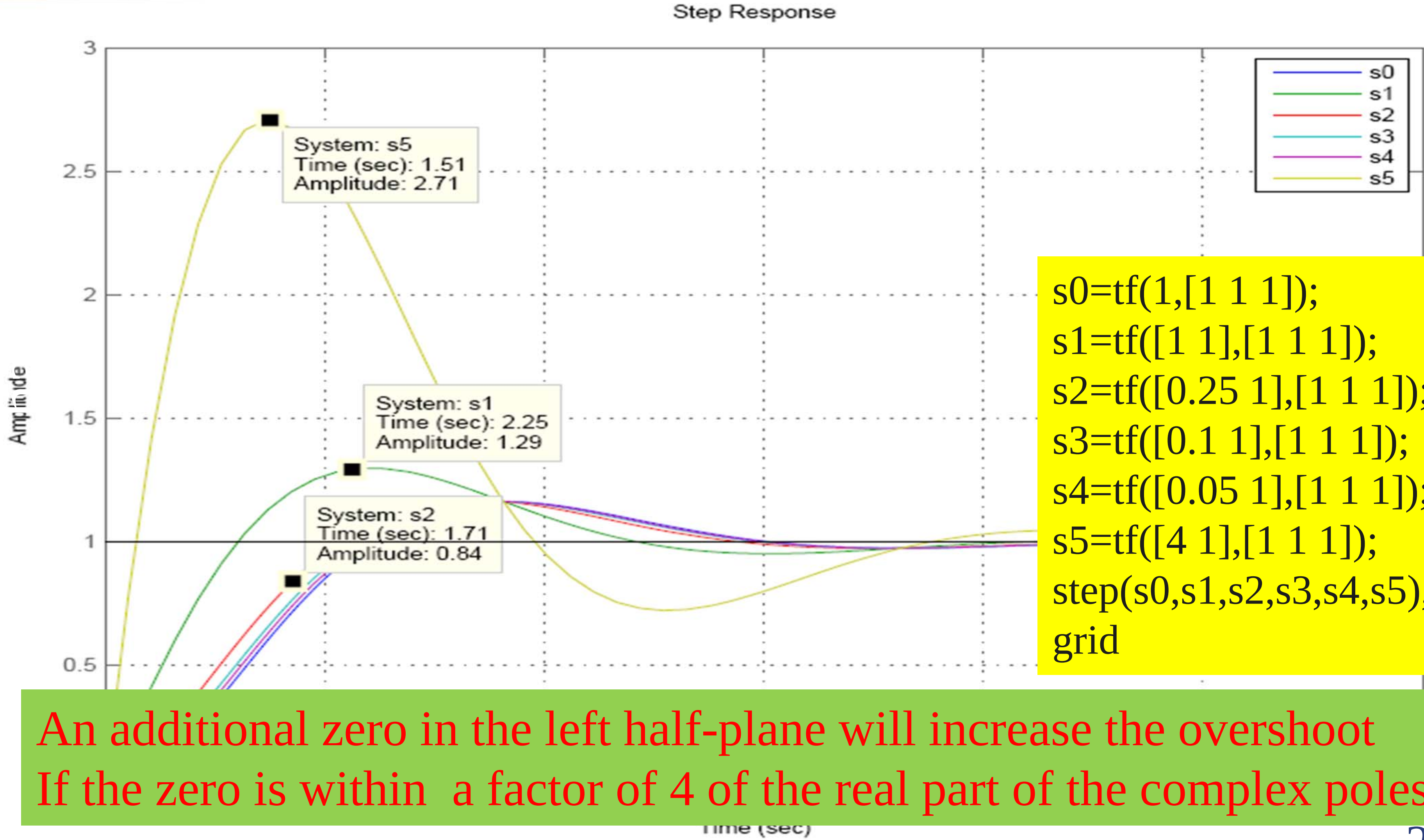
pole: $-\dfrac{1}{\tau}$, time constant: τ,

$G(s) = \dfrac{\omega_n^2}{s^2+2\xi\omega_n+\omega_n^2}$, assume $\omega_n > 0$, $\xi \geq 0$

poles: $p_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

How about the effects of zero(s) to system performance?

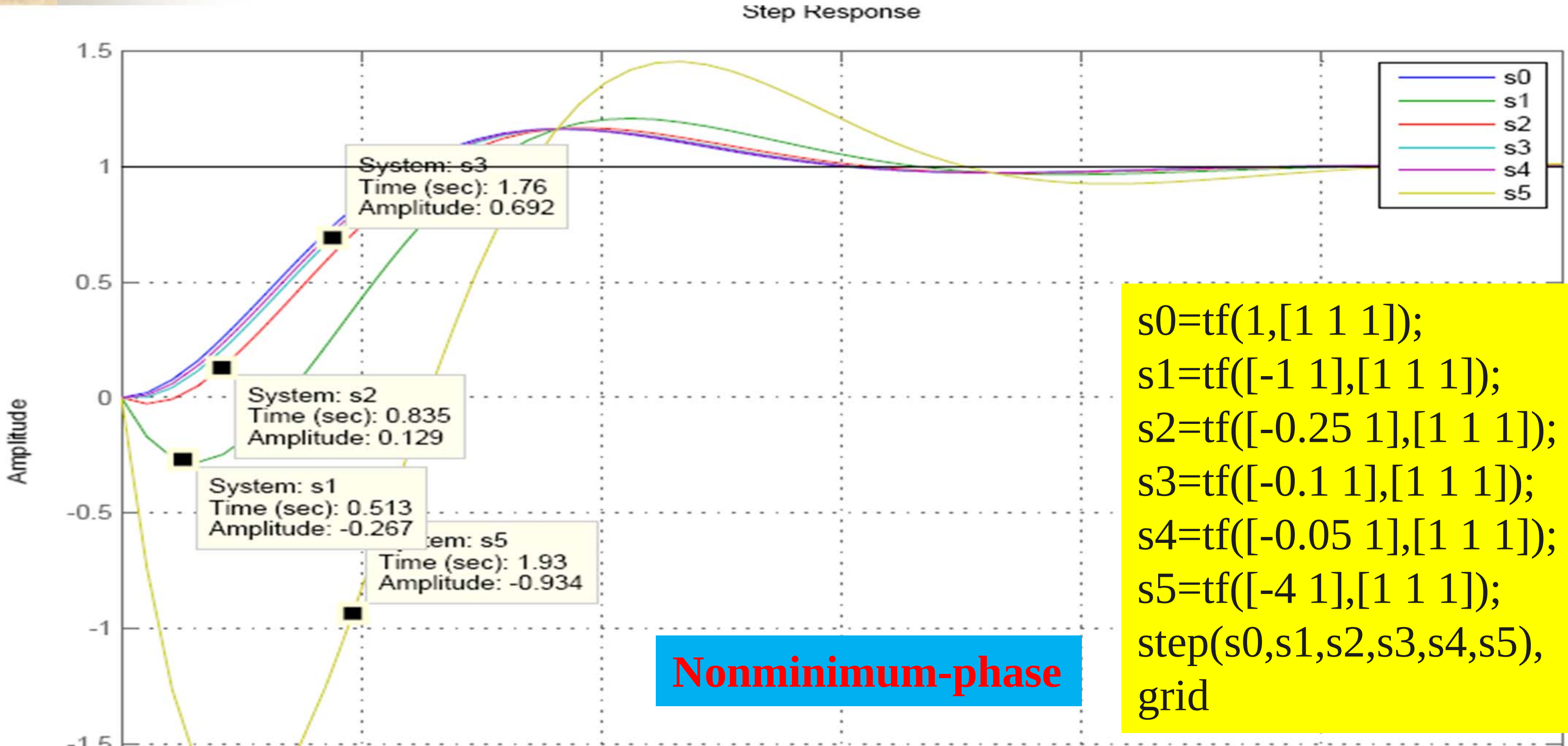
Effect of Zero in the Left Half s-Plane




```
s0=tf(1,[1 1 1]);
s1=tf([1 1],[1 1 1]);
s2=tf([0.25 1],[1 1 1]);
s3=tf([0.1 1],[1 1 1]);
s4=tf([0.05 1],[1 1 1]);
s5=tf([4 1],[1 1 1]);
step(s0,s1,s2,s3,s4,s5),
grid
```

An additional zero in the left half-plane will increase the overshoot
If the zero is within a factor of 4 of the real part of the complex poles

Effect of Zero in the Right Half s-Plane

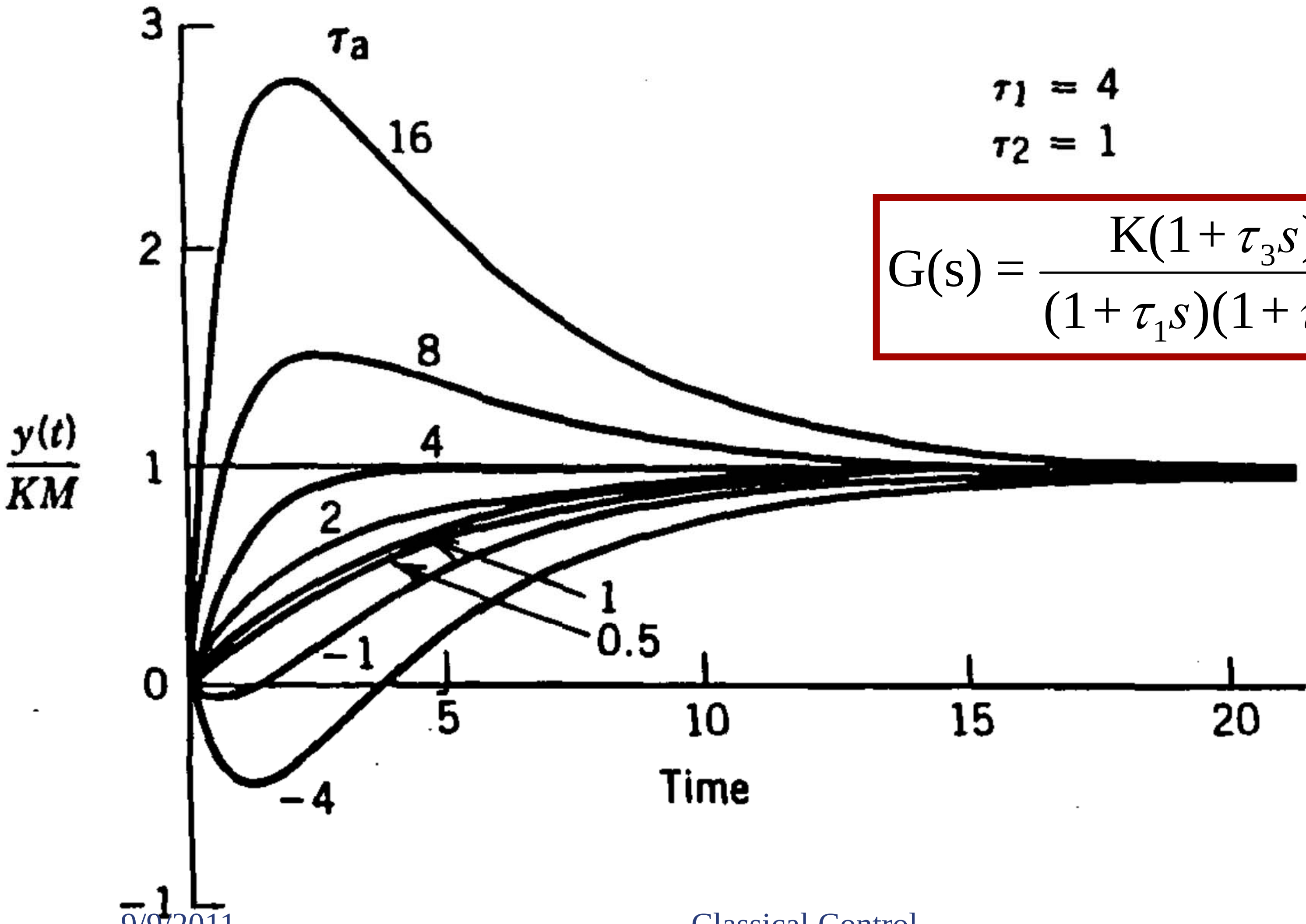


Nonminimum-phase

```
s0=tf(1,[1 1 1]);
s1=tf([-1 1],[1 1 1]);
s2=tf([-0.25 1],[1 1 1]);
s3=tf([-0.1 1],[1 1 1]);
s4=tf([-0.05 1],[1 1 1]);
s5=tf([-4 1],[1 1 1]);
step(s0,s1,s2,s3,s4,s5),
grid
```

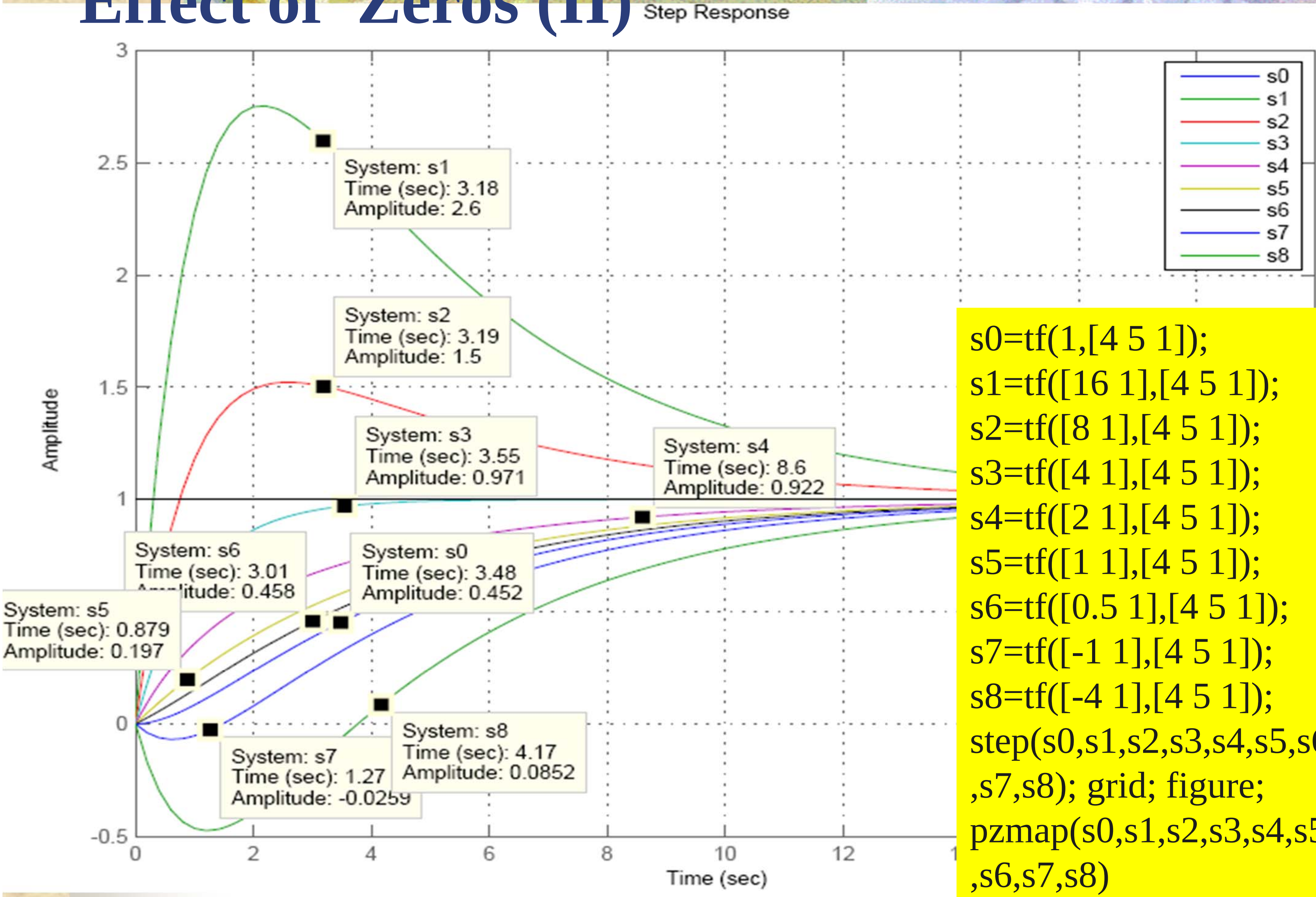
An additional zero in the right half-plane will depress the overshoot and may cause the step response to start out in the wrong direction

Effect of Zeros (I)



$$G(s) = \frac{K(1+\tau_3 s)}{(1+\tau_1 s)(1+\tau_2 s)}$$

Effect of Zeros (II)




```
s0=tf(1,[4 5 1]);
s1=tf([16 1],[4 5 1]);
s2=tf([8 1],[4 5 1]);
s3=tf([4 1],[4 5 1]);
s4=tf([2 1],[4 5 1]);
s5=tf([1 1],[4 5 1]);
s6=tf([0.5 1],[4 5 1]);
s7=tf([-1 1],[4 5 1]);
s8=tf([-4 1],[4 5 1]);
step(s0,s1,s2,s3,s4,s5,s6
,s7,s8); grid; figure;
pzmap(s0,s1,s2,s3,s4,s5
,s6,s7,s8)
```

Execise Five

- Determine system's stability on slide p.13
- Determine your project system's stability, see slide p.17
- Steady-state error analysis, see slide p.25