MM8 Frequency Response Analysis (I) – Bode Plot

Readings:

- Section 6.1 (frequency response, page338-358);
- Section 6.4 (stability margins, page 375-379);
- Section 6.6 (closed-loop frequency response, page 388-389);

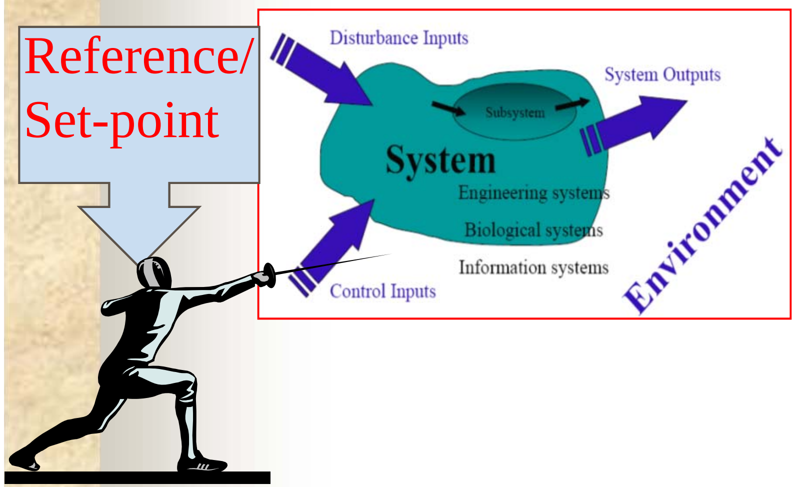
What have we talked about lecture in MM7

Some **practical issues** when developing a PID controler:

- **Integral windup & Anti-windup methods**
- **Derivertive kick**
- **When to use which controller?**
- **Operational Amplifier Implementation**
- **Other tuning methods**

MM6: Control Objectives

Control is a process of causing a system (output) variable to conform to some desired status/value (**MM1**)

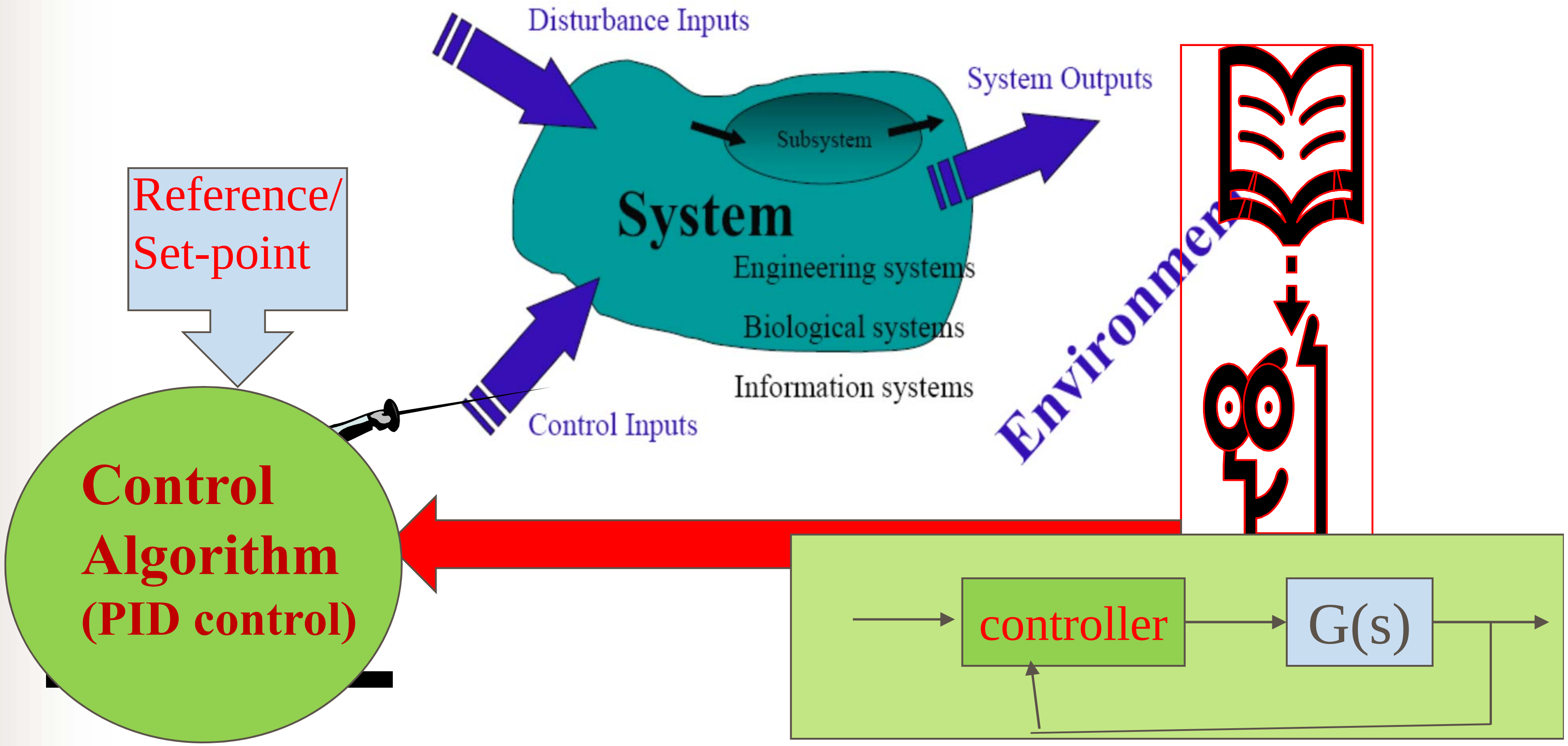


Control Objectives

- **Stable (MM5)**
- **Quick responding (MM3, 4)**
- **Adequate disturbance rejection**
- **Insensitive to model & measurement errors**
- **Avoids excessive control action**
- **Suitable for a wide range of operating conditions**

MM4:Control Strategy: Feedback Control

- **Closed-loop Control:** A control process which utilizes the feedback mechanism, i.e., the output(s) does have effect upon the control input(s)

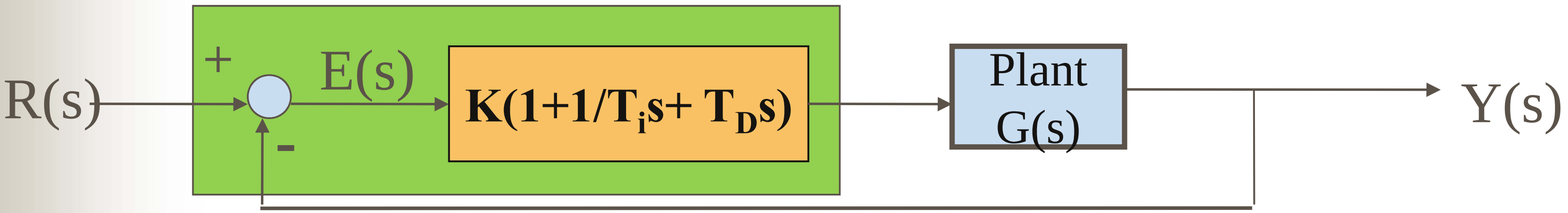


MM7: Controller Synthesis (Time Domain)

Time-domain techniques can be classified into two groups:

- Criteria based on a few points in the response
  (settling time, overshoot, rise time, decay ratio, settling time)
  for example, root locus, Ziegler-Nicols, Cohen-coon

- Criteria based on the entire response, or integral criteria
  for example, IAE, ISE, ITAE for PID tunings



Goals for this lecture (MM8)

Essentials for **frequency domain** design methods – **Bode plot**

- **Bode plot analysis**
  - **How to get a Bode plot**
  - **What we can gain from Bode plot**
- How to use bode plot for design purpose
  - Stability margins (Gain margin and phase margin)
  - Transient performance
  - Steady-state performance
- Matlab functions: bode(), margin()

Frequency Domain Analysis

Frequency response

$$u(t) = U_0 \sin(\omega_0 t)$$

$$y(t) = U_0 A \sin(\omega_0 t + \theta)$$

$$A = |G(s)|_{s=j\omega_0} = \sqrt{\{\mathrm{Re}[G(j\omega_0)]\}^2 + \{\mathrm{Im}[G(j\omega_0)]\}^2}$$

$$\theta = \angle G(s)|_{s=j\omega_0} = \tan^{-1} \frac{\mathrm{Im}[G(j\omega_0)]}{\mathrm{Re}[G(j\omega_0)]}$$

Frequency Response

- The frequency response $\mathbf{G(j\Omega)\ (=G(s)|_{s=j\Omega})}$ is a representation of the system's response to sinusoidal inputs at varying frequencies

$$G(j\Omega) = |G(j\Omega)|\, e^{\angle G(j\Omega)},$$

- Input $x(n)$ and output $y(n)$ relationship

$$|Y(j\Omega)| = |H(j\Omega)|\, |X(j\Omega)|$$

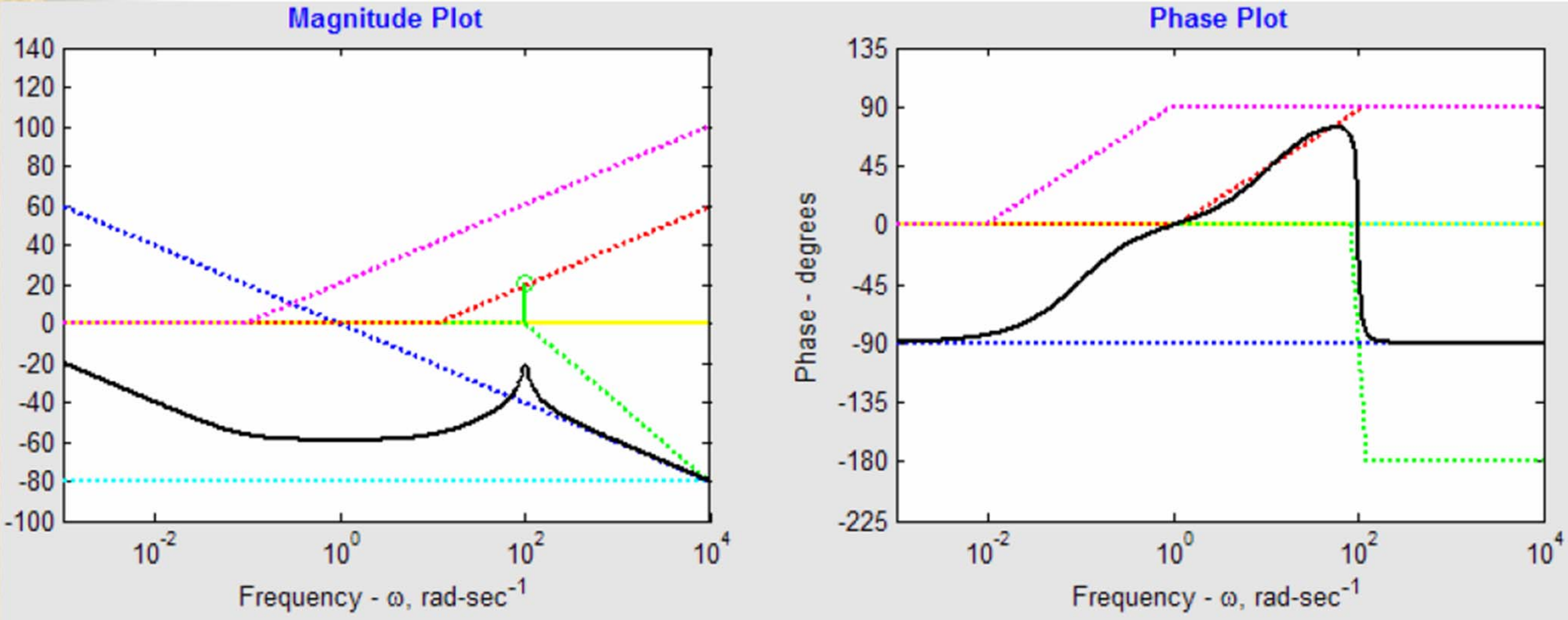
$$\angle Y(j\Omega) = \angle H(j\Omega) + \angle X(j\Omega)$$

- The frequency response of a system can be viewed
  - via the **Bode plot** (H.W. Bode 1932-1942)
  - via the **Nyquist diagram**

Bode plot – **What's that?**

- Bode plot is a graphic representation of the magnitude (logarithmic scale – Decibel (dB)) and phase (linear scale) of the system's frequency response

$$\mathbf{H(j\Omega) = |H(j\Omega)|\, e^{\sphericalangle H(j\Omega)}}$$



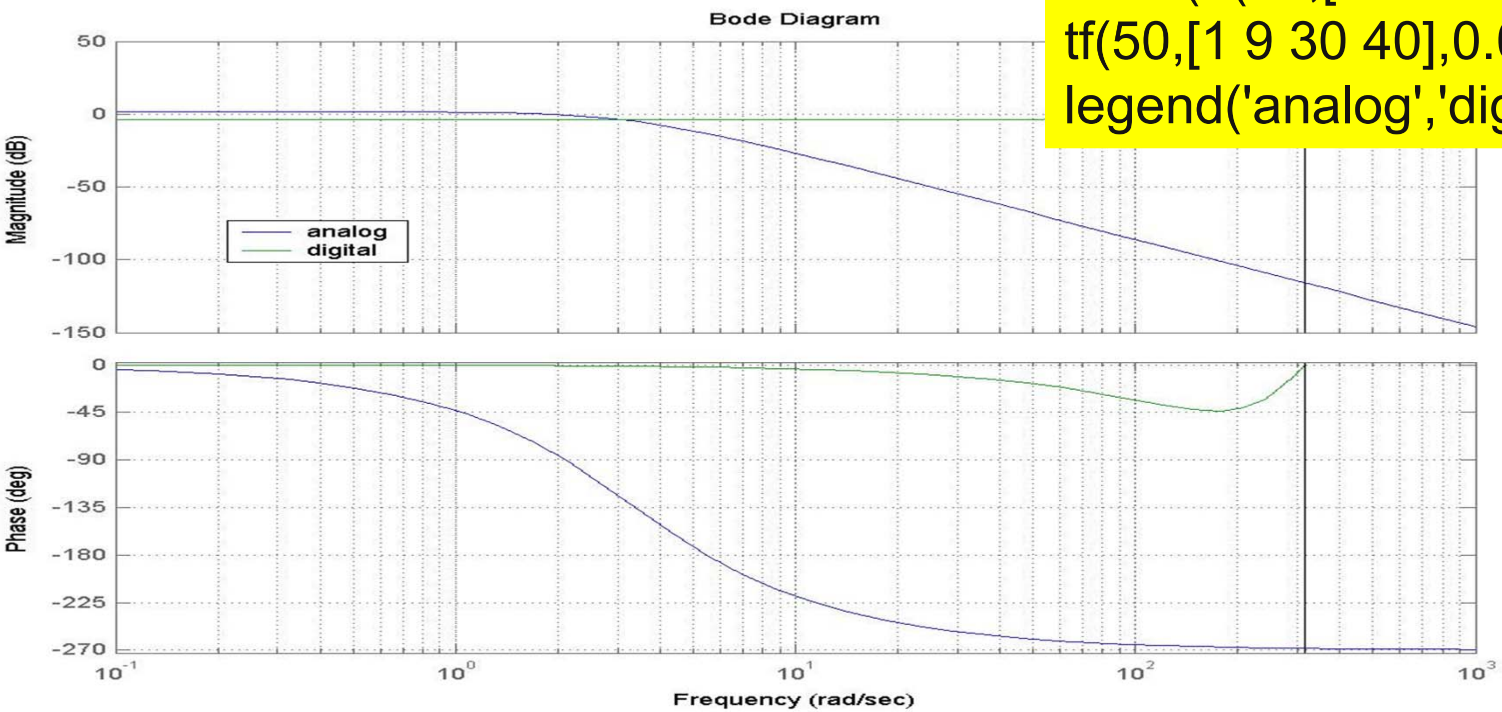
Bode plot – **Why Use It?**

- Bode plots of systems in series simply add
- The **phase-gain relationship** has a unique relationship for any stable monimum-phase system
- A much wider range of the system behavior – from low to high frequency – can be displayed on a single plot;
- Bode plot can be determined **experimentally**
- **Dynamic compensator** design can be based entirely on Bode plots

Bode plot – **How to get it?**

- Experimental method
- Manual drawing (based on TF information)
- Comercial software: Matlab – **bode()**

Example:
bode(tf(50,[1 9 30 40]),
tf(50,[1 9 30 40],0.01));
legend('analog','digital‘)



Bode plot – **Why important to hand-plot?**

- To deal with simple problems
- To check computer results for more complicated cases
- To deduce stability
- To deduce the form of the needed dynamic compensators
- To interprete frequency-response data generated experimentally

Bode plot – **Hand-Plot Rules (I)**

Term	Magnitude	Phase
Constant: K	$20 \cdot \log_{10}(\lvert K \rvert)$	$K > 0: \ 0^\circ$ $K < 0: \ \pm 180^\circ$
Real Pole: $\dfrac{1}{\frac{s}{\omega_0}+1}$	• Low freq. asymptote at 0 dB • High freq. asymptote at -20 dB/dec • Connect lines at ω_0	• Low freq. asymptote at 0° • High freq. asymptote at -90° • Connect with straight line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$
Real Zero*: $\frac{s}{\omega_0}+1$	• Low freq. asymptote at 0 dB • High freq. asymptote at +20 dB/dec. • Connect lines at ω_0	• Low freq. asymptote at 0° • High freq. asymptote at $+90^\circ$ • Connect with line from $0.1 \cdot \omega_0$ to $10 \cdot \omega_0$
Pole at Origin: $\frac{1}{s}$	-20 dB/dec; through 0 dB at $\omega=1$	-90°
Zero at Origin*: s	+20 dB/dec; through 0 dB at $\omega=1$	$+90^\circ$

Rough estimation of peak amplitude: **$|G(j\Omega)|=1/2\xi$ at $\Omega=\omega_n$**

Bode plot – **Hand-Plot Rules (II)**

Underdamped Poles: $\dfrac{1}{\left(\dfrac{s}{\omega_0}\right)^2+2\zeta\left(\dfrac{s}{\omega_0}\right)+1}$	• Low freq. asymptote at 0 dB • High freq. asymptote at -40 dB/dec. • Draw peak† at freq. $\omega_r=\omega_0\sqrt{1-2\zeta^2}$ with amplitude • $H(j\omega_r)=-20\cdot\log_{10}\left(2\zeta\sqrt{1-\zeta^2}\right)$ • Connect lines	• Low freq. asymptote at 0° • High freq. asymptote at -180° • Connect with straight line from ‡ $\omega=\omega_0\dfrac{\log_{10}\left(\frac{2}{\zeta}\right)}{2}$ to $\omega=\omega_0\dfrac{2}{\log_{10}\left(\frac{2}{\zeta}\right)}$
Underdamped Zeros*: $\left(\dfrac{s}{\omega_0}\right)^2+2\zeta\left(\dfrac{s}{\omega_0}\right)+1$	• Draw low freq. asymptote at 0 dB • Draw high freq. asymptote at +40 dB/dec. • Draw dip† at freq. $\omega_r=\dfrac{\omega_0}{\sqrt{1-2\zeta^2}}$ with amplitude $H(j\omega_r)=+20\cdot\log_{10}\left(2\zeta\sqrt{1-\zeta^2}\right)$ • Connect lines	• Low freq. asymptote at 0° • Draw high freq. asymptote at $+180^\circ$ • Connect with a straight line from ‡ $\omega=\omega_0\dfrac{\log_{10}\left(\frac{2}{\zeta}\right)}{2}$ to $\omega=\omega_0\dfrac{2}{\log_{10}\left(\frac{2}{\zeta}\right)}$

Notes:

* Rules for drawing zeros create the mirror image (around 0 dB, or 0°) of those for a pole with the same ω_0.

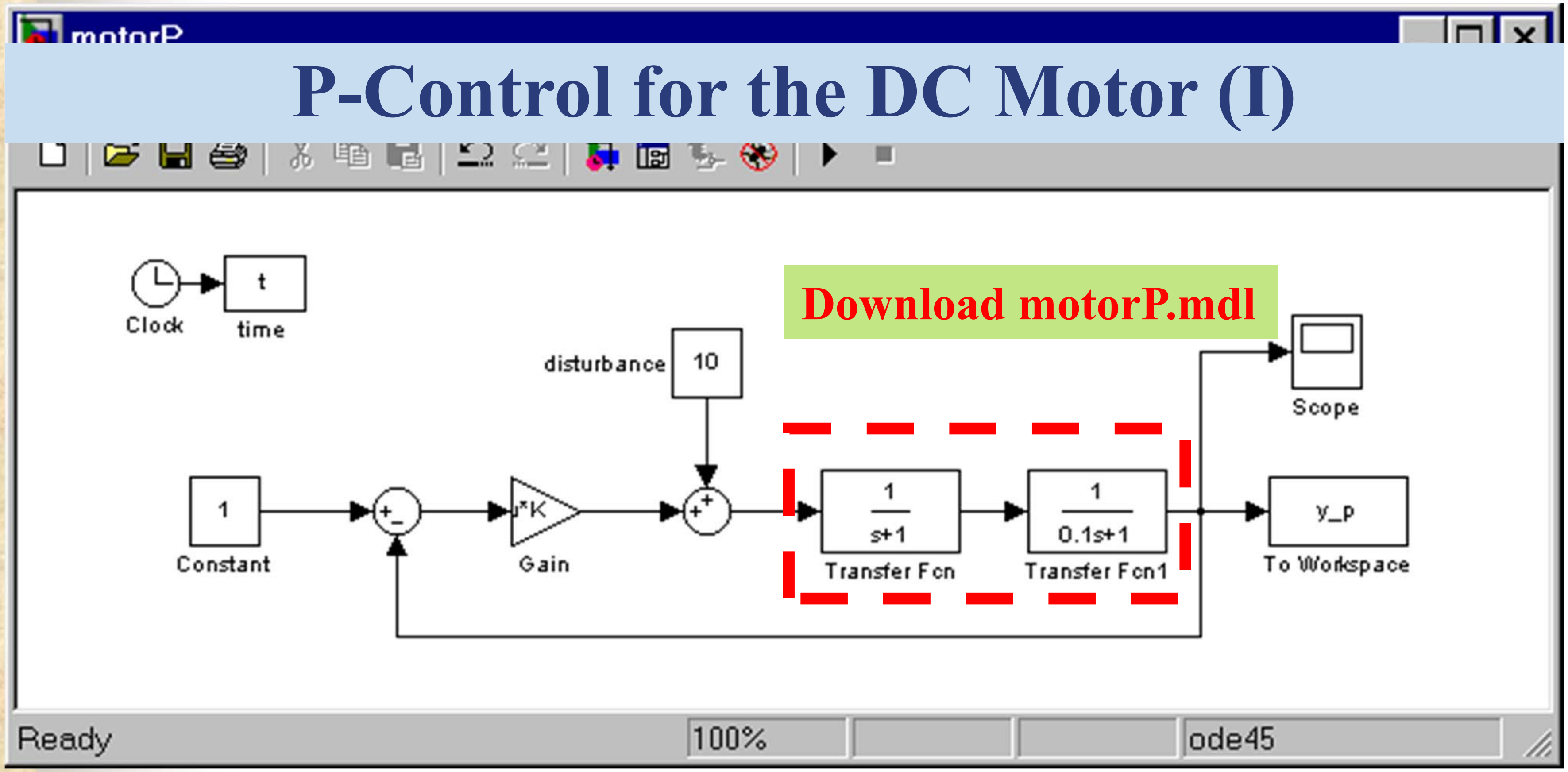
† For underdamped poles and zeros peak exists only for $0<\zeta<0.707=\frac{1}{\sqrt{2}}$ and peak freq. is typically very near ω_0.

‡ For underdamped poles and zeros If $\zeta<0.02$ draw phase vertically from 0 to -180 degrees at ω_0

For n^{th} order pole or zero make asymptotes, peaks and slopes n times higher than shown (i.e., second order asymptote is -40 dB/dec, and phase goes from 0 to -180°). Don't change frequencies, only plot values and slopes.

Execise: Hand-plot Bode Plot

- Sketch the Bode plot of the DC-motor model (from MM6)

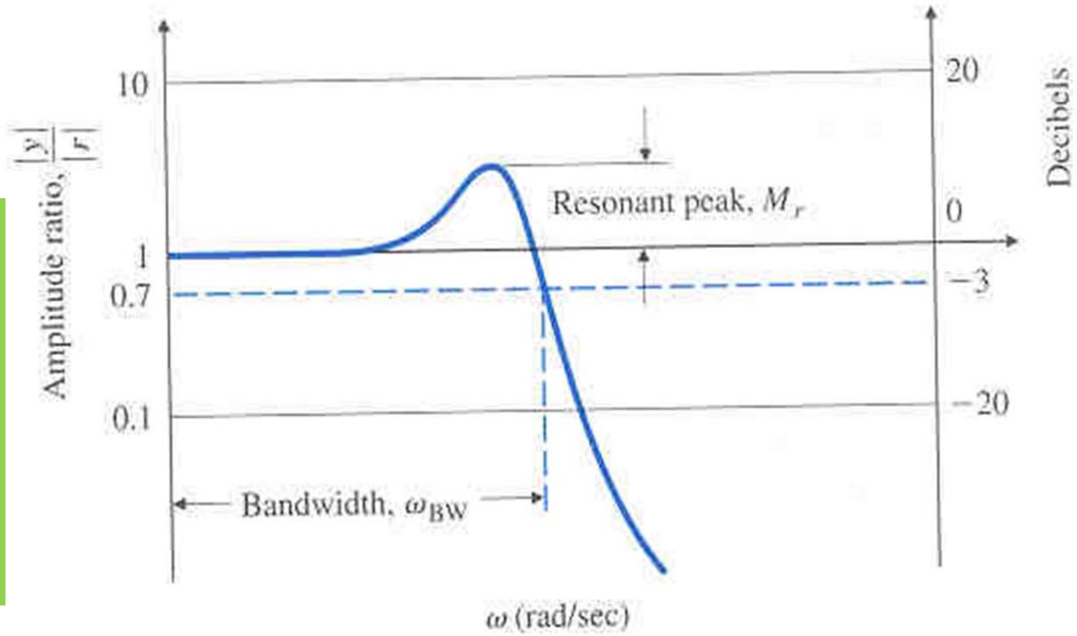


Bode Plot Analysis– **What can we know? (I)**

Bandwidth

- The maximum frequency at which the output of the system will track an input sinusoid in a satisfactory manner
- Measure of the speed of response (–3dB frequency)

Time domain:
rise time, settling time
S-plane:
Dominant pole(s)
natural frequency



Bandwidth Frequency

- Sinusoidal inputs with frequency less than Wbw are tracked "reasonably well" by the system. Sinusoidal inputs with frequency greater than Wbw are attenuated (in magnitude) by a factor of 0.707 or greater (and are also shifted in phase).

- **Example:**

```
bode (1, [1 0.5 1 ]); grid
```

```
w= 0.3;
num = 1; den = [1 0.5 1 ];
t=0:0.1:100;
u = sin(w*t);
[y,x] = lsim(num,den,u,t);
plot(t,y,t,u)
axis([50,100,-2,2])
legend('output','input')
```

```
w = 3;
num = 1; den = [1 0.5 1 ];
t=0:0.1:100;
u = sin(w*t);
[y,x] = lsim(num,den,u,t);
plot(t,y,t,u)
axis([90, 100, -1, 1])
legend('output','input')
```

Bode Plot Analysis– **What can we know? (II)**

Resonant Peak

- The maximum value of the frequency response magnitude
- Is relevant to dampling ratio **ξ**
- Rough estimation of peak amplitude: **|G(jΩ)|=1/2ξ at Ω=ω_n**

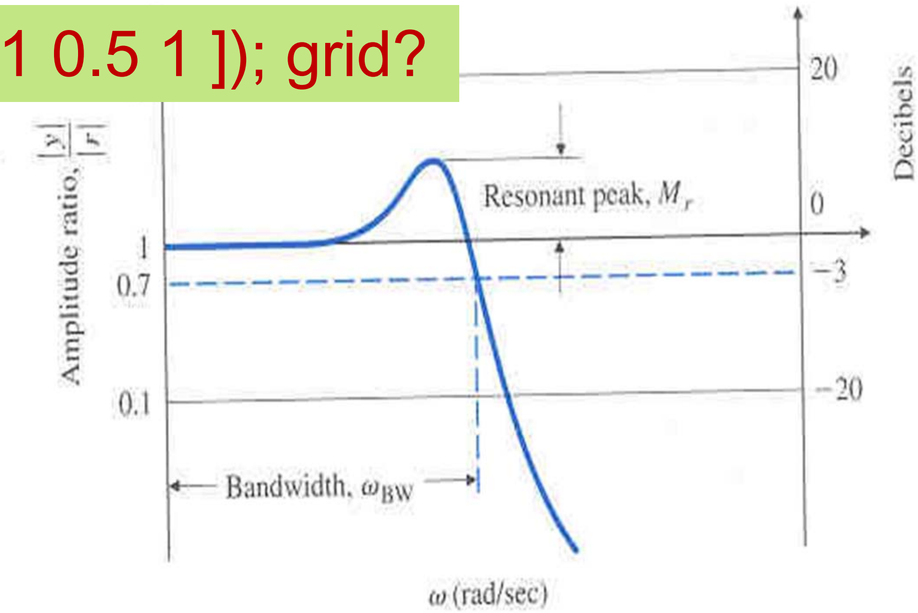
Is that true with bode (1, [1 0.5 1]); grid?

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$h(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\sigma t} \sin(\omega_d t) 1(t)$$

ζ : damping ratio $\quad \omega_n$: natural frequency

$$\sigma = \zeta\omega_n, \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

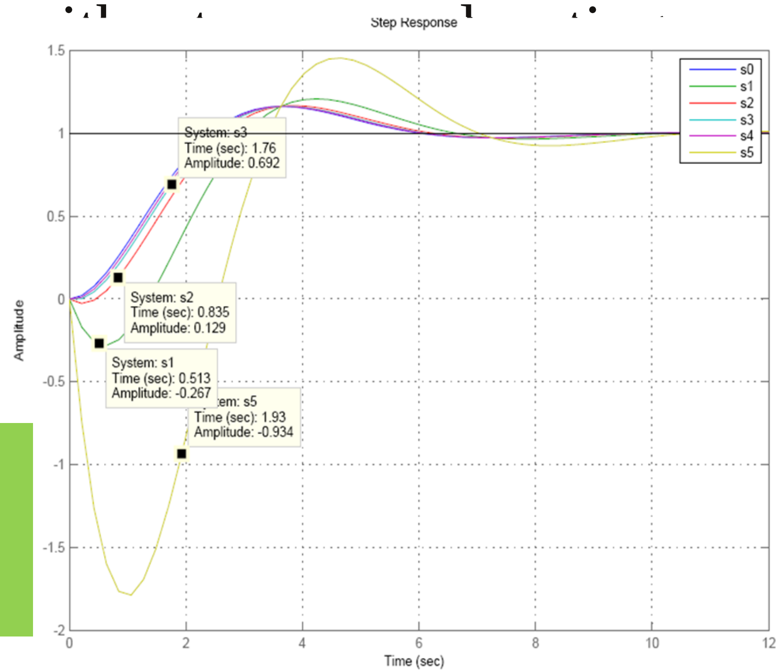


Bode Plot Analysis– **What can we know? (III)**

$$G(s) = \frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

- **Stable** vs **nonstable** systems
- **Minimum-phase system:** system
  the right half-s-plane
- **Nonminimum-phase system**

  (effect of additional zero- **MM5)**

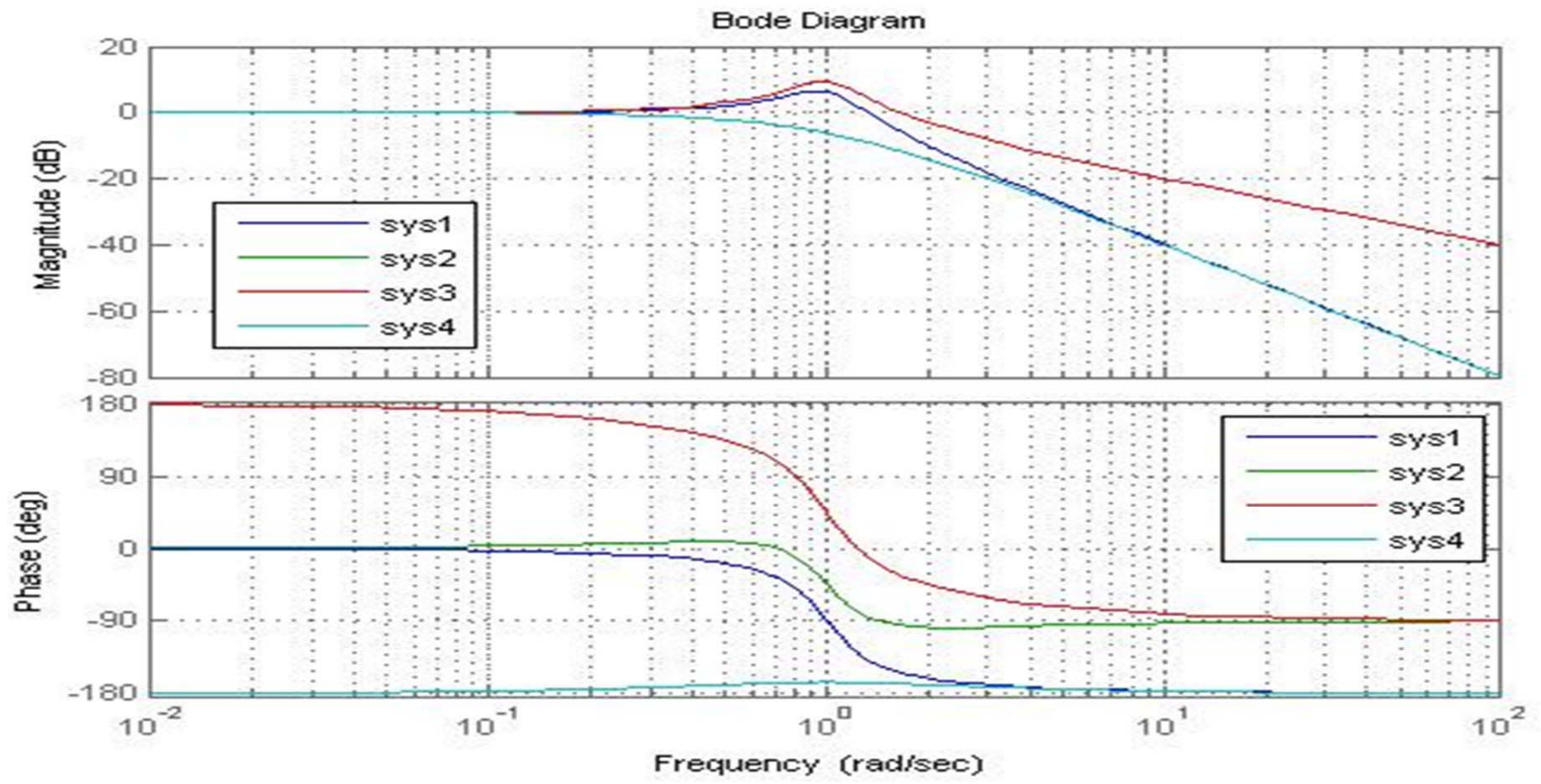


An additional zero in the right half-plane will depress the overshoot and may cause step response to start out in the wrong direction

sys1=tf(1,[1 0.5 1]);
sys2=tf([1 1],[1 0.5 1]);
sys3=tf([1 -1],[1 0.5 1]);
sys4=tf([1],[1 0.5 -1]);

Execise: Determine system features

- Determine the following four systems' features (stable or unstable, minimum-phase or nonminimum-phase)

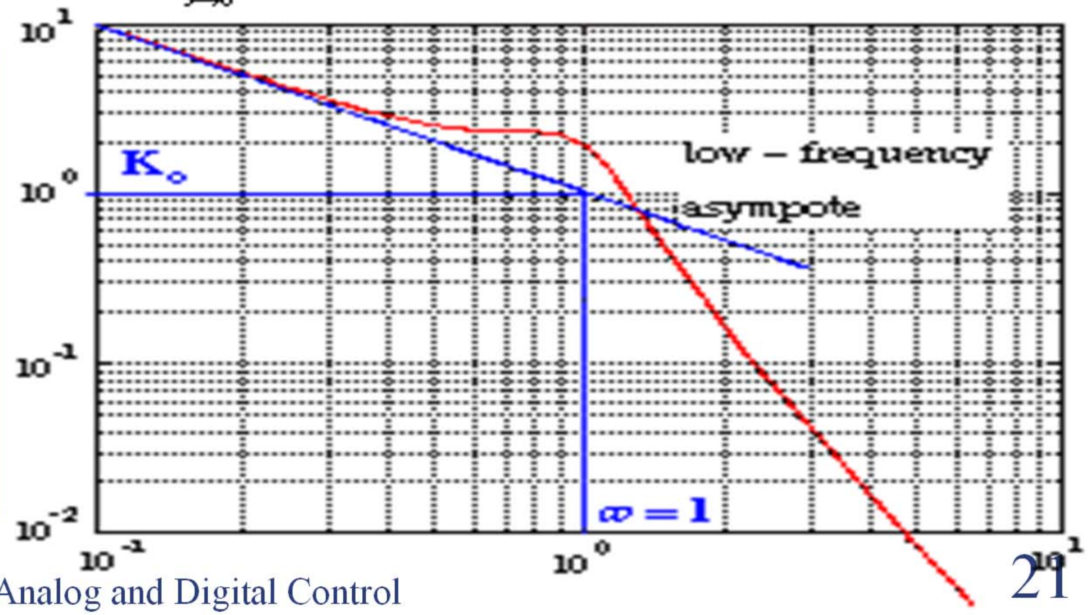


Bode Plot – **What can we can know? (IV)**

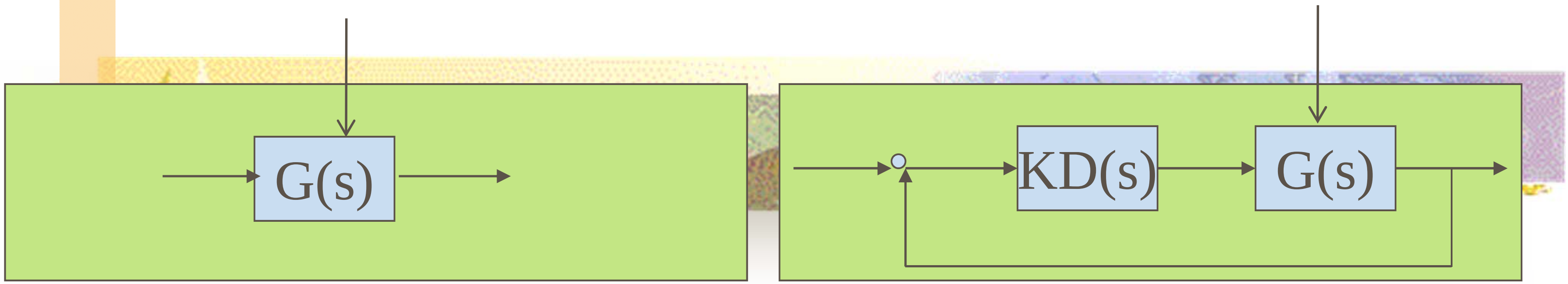
Types of the system (Steady State Error)

- Type-0 system: $e(\infty) = \dfrac{1}{1+\lim\limits_{s\to 0} G(s)} = \dfrac{1}{1+K_p} \Rightarrow K_p = \lim\limits_{s\to 0} G(s)$
- Type-1 system: $e(\infty) = \dfrac{1}{\lim\limits_{s\to 0} sG(s)} = \dfrac{1}{K_v} \Rightarrow K_v = \lim\limits_{s\to 0} sG(s)$
- Type-2 system: $e(\infty) = \dfrac{1}{\lim\limits_{s\to 0} s^2G(s)} = \dfrac{1}{K_a} \Rightarrow K_a = \lim\limits_{s\to 0} s^2G(s)$

The constant (Kp, Kv, or Ka) are located at the intersection of the low frequency asymptote with the w=1 line. Just extend the low frequency line to the w=1 line. The magnitude at this point is the constant.



Can we further use Bode plot for Control Design Purpose?



For example,

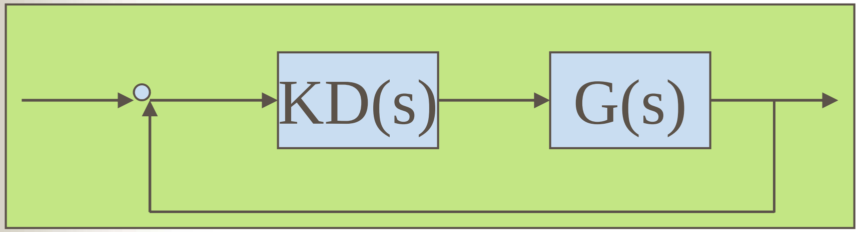
Setup the relationship of the open-loop characteristics to the closed-loop characteristics, e.g., **stability?** Bandwidth? Overshoot? Steady-state error?

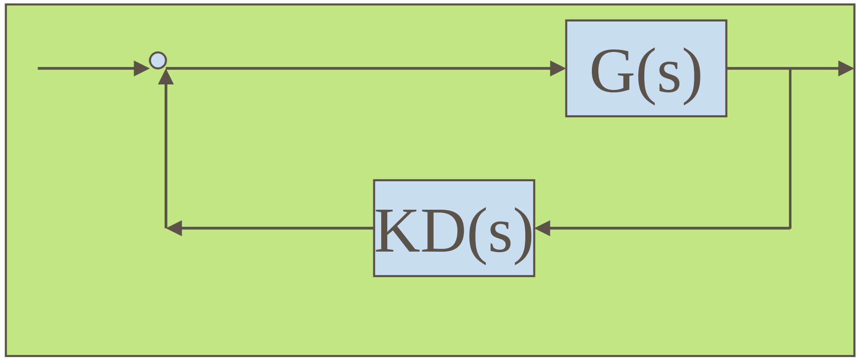
Open-Loop Transfer Function

- **Motivation**

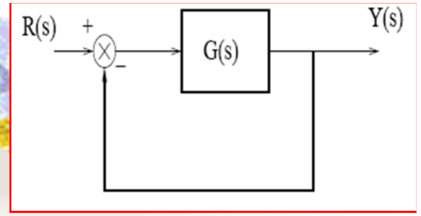
  Predict the closed-loop system's properties using the open-loop system's frequency response

- **Open-loop TF (Loop gain) :** $L(s)=KD(s)G(s)$



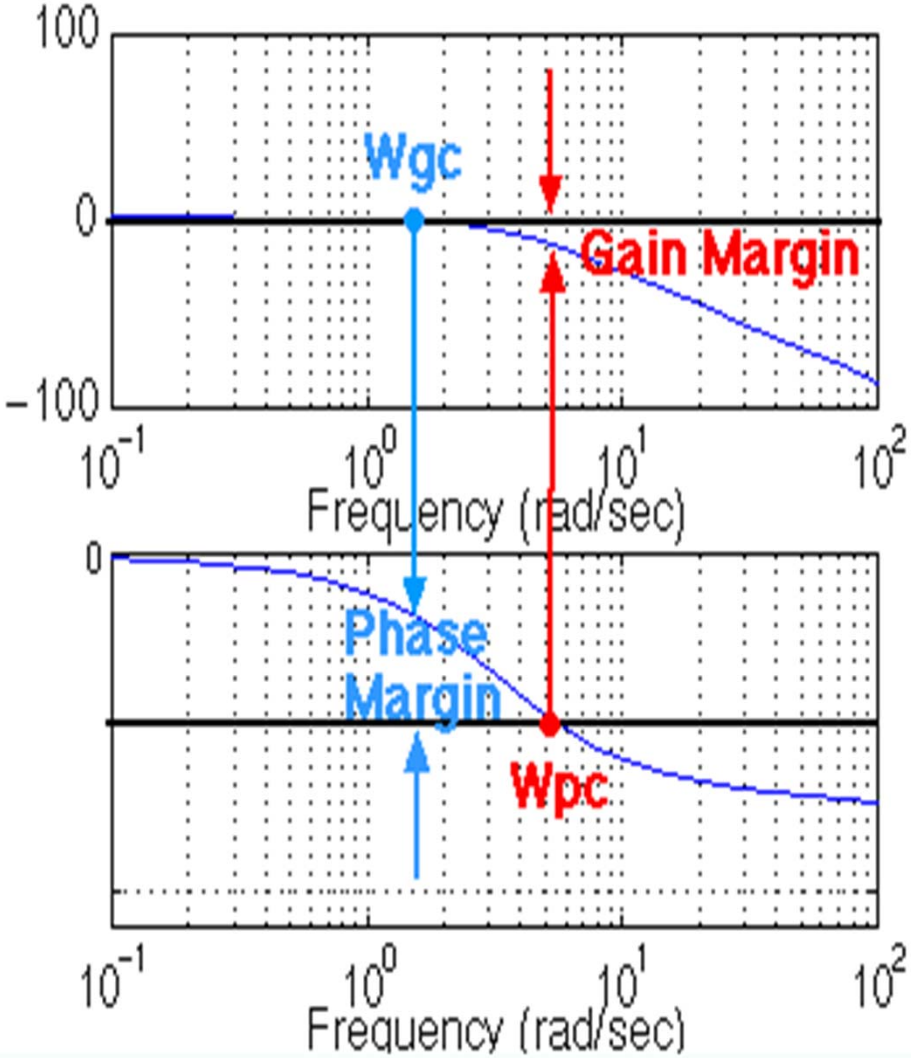


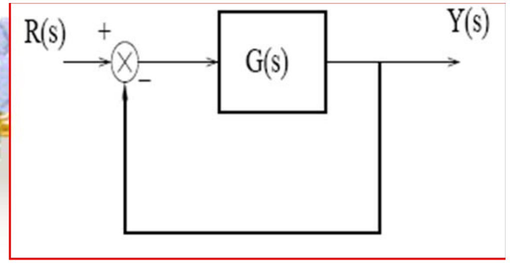
- **Closed-loop:** $G_{cl}(s)=L(s)/(1+L(s))$, or $G_{cl}(s)=G(s)/(1+L(s))$



Definition of Phase Margin (PM)

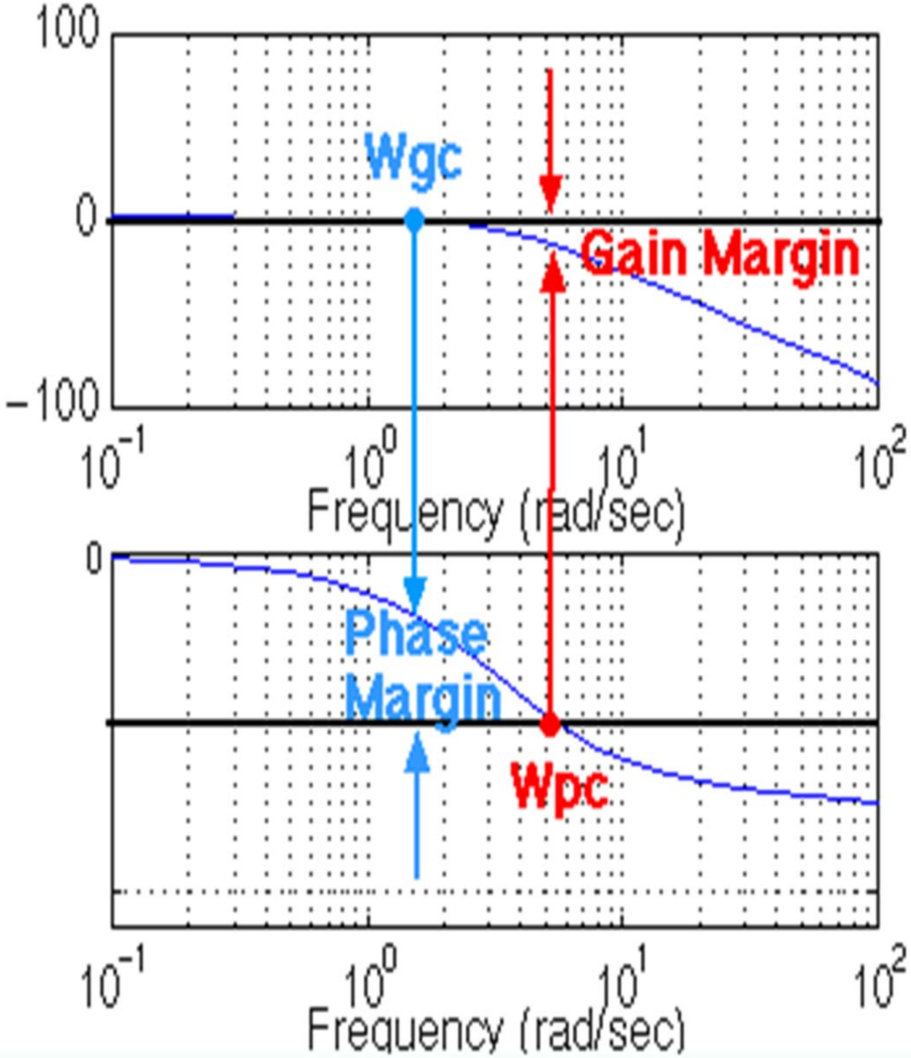
- Bode plot of the open-loop TF
- The **phase margin** is the difference in phase between the phase curve and -180 deg at the point corresponding to the frequency that gives us a gain of 0dB (the **gain cross over frequency**, Wgc).

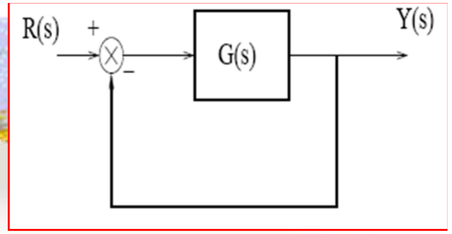




Meanings of Phase Margin

- The **phase margin** reflects the change in open loop phase shift required to make a closed loop system **unstable**.
- The phase margin also measures the system's tolerance to **time delay**
- Robustness

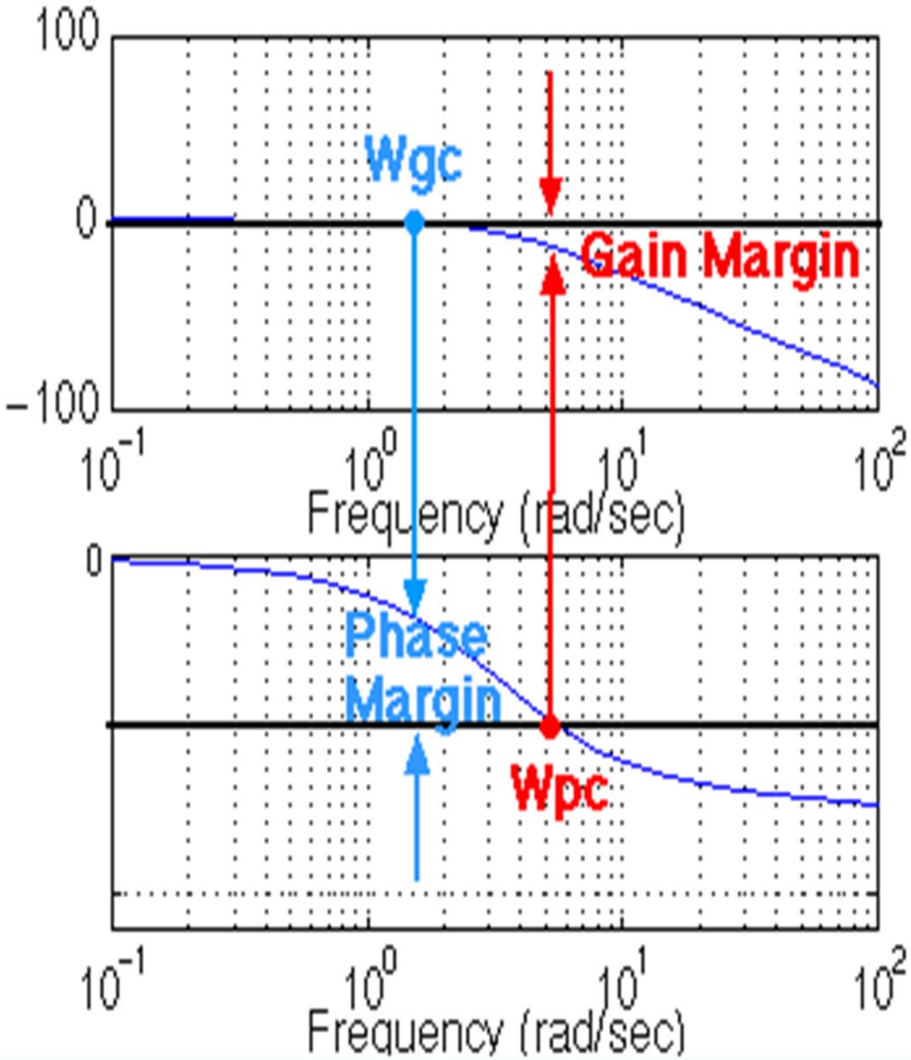


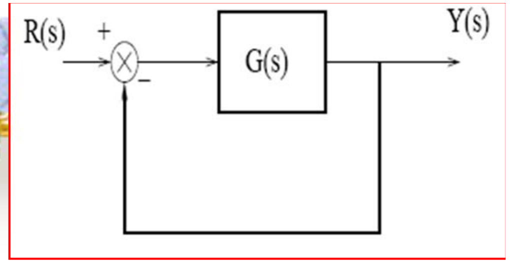


Definition of Gain Margin (GM)

- Bode plot of the open-loop TF

- The **gain margin** is the difference between the magnitude curve and 0dB at the point corresponding to the frequency that gives us a phase of -180 deg (the **phase cross over frequency**, Wpc).

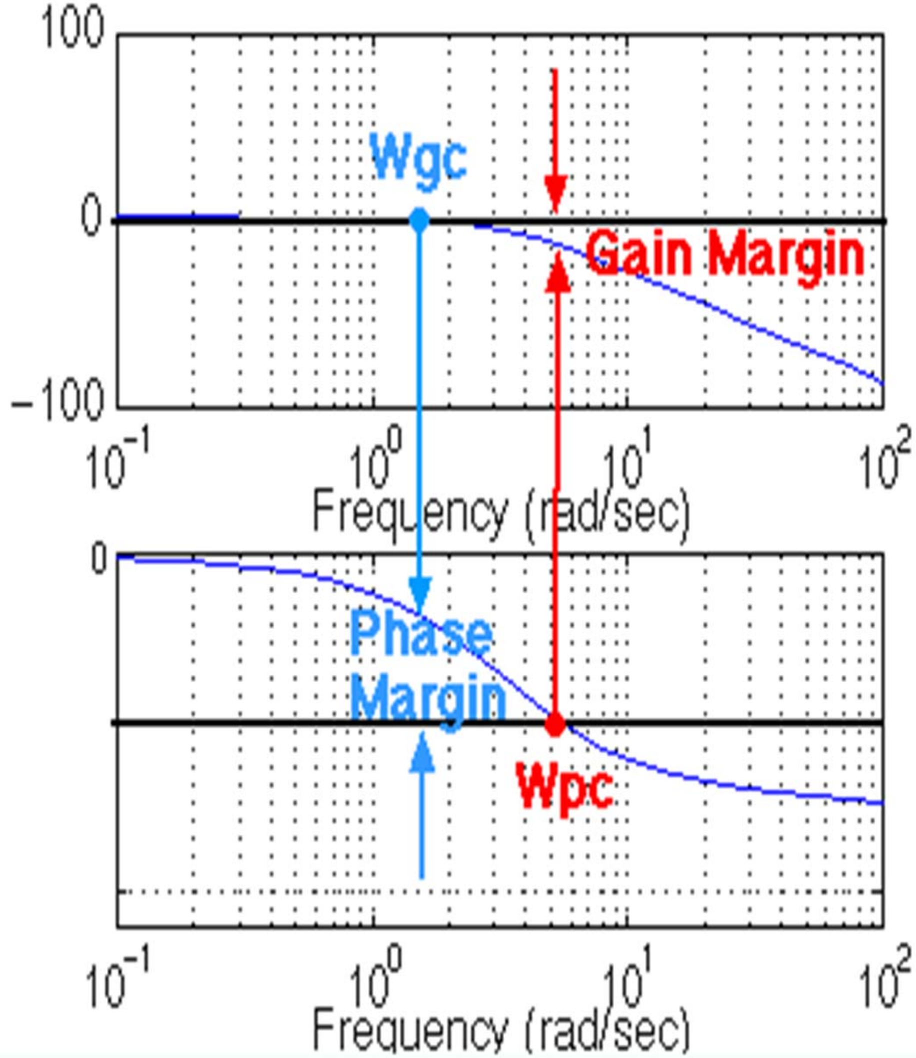


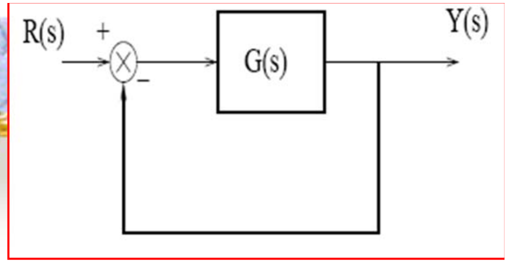


Meanings of of Gain Margin

- The **gain margin** reflects the change in open loop gain required to make the system **unstable**.

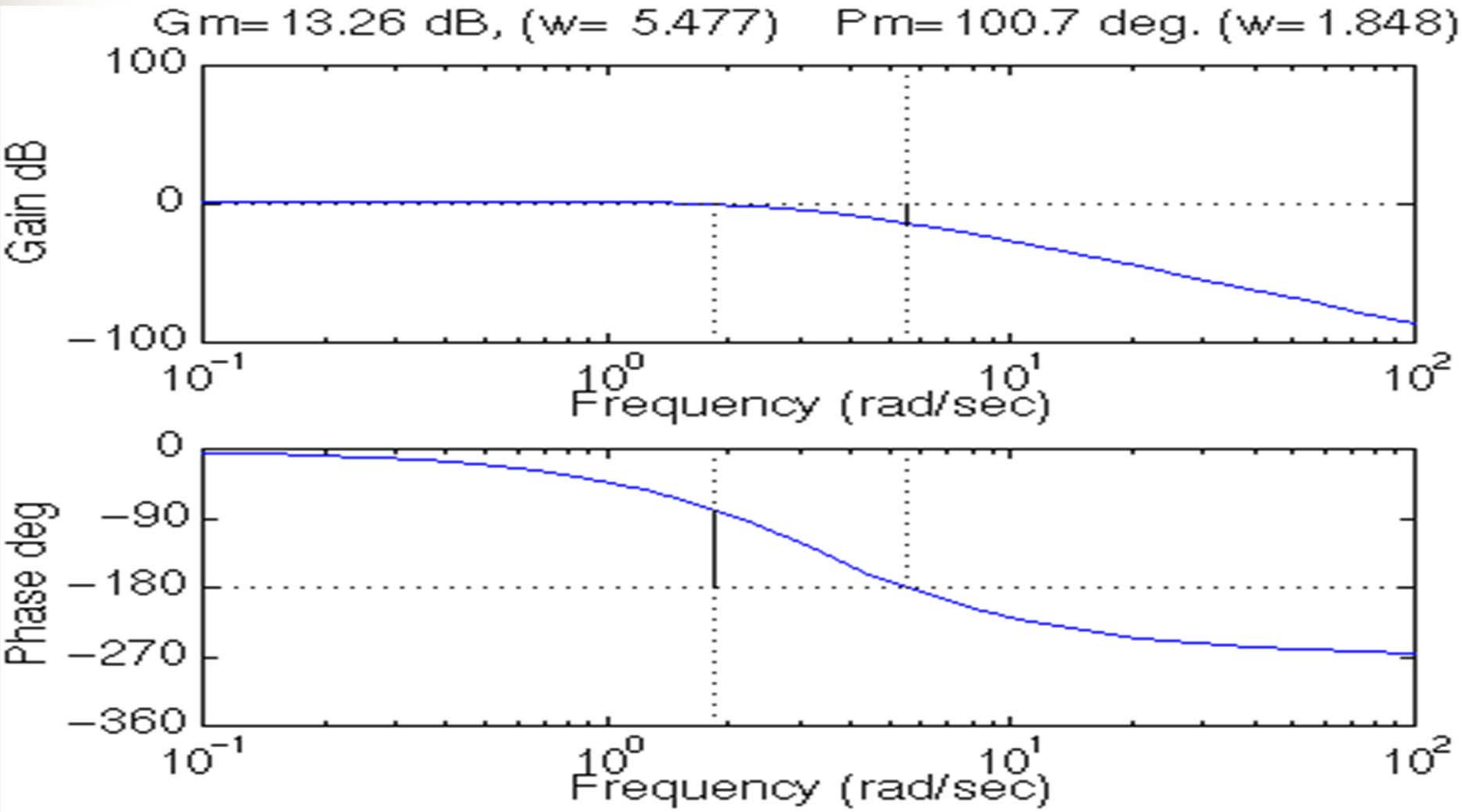
- Systems with greater gain margins can withstand greater **changes in system parameters** before becoming unstable in closed loop.

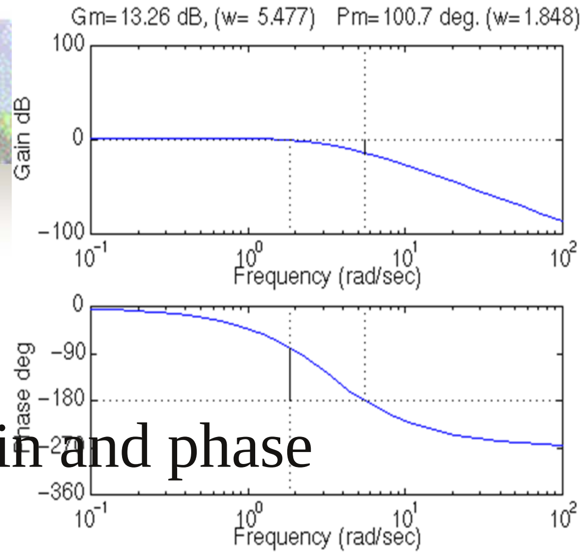




How to Find GM &PM?

- Manual/CAD bode plot: margin(50,[1 9 30 40])

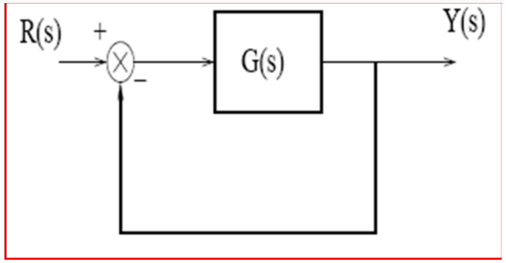




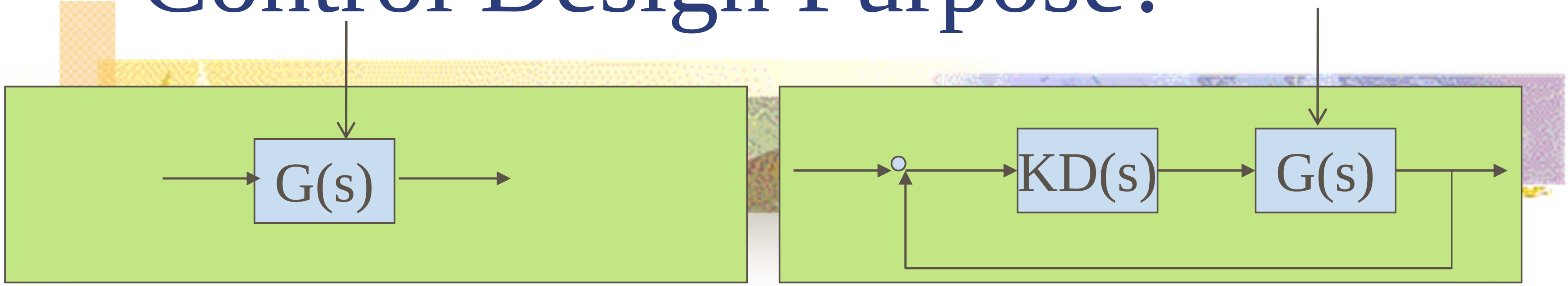
How to Use GM &PM?

- A **stable** system should have positive gain and phase margins
- Systems with greater gain margins can withstand greater **changes in system parameters** before becoming unstable in closed loop
- The phase margin also measures the system's tolerance to **time delay**
- Adding gain only shifts the magnitude plot up. Finding the phase margin is simply the matter of finding the new cross-over frequency and reading off the phase margin

```
margin(50,[1 9 30 40])
margin(100*50,[1 9 30 40])
```

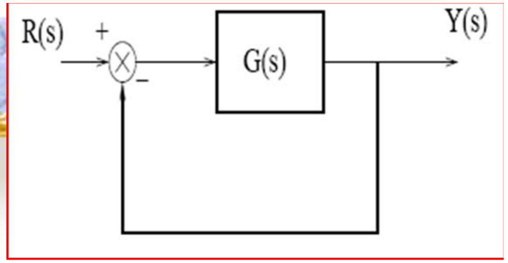


Can we further use Bode plot for Control Design Purpose?



For example,

Setup the relationship of the open-loop characteristics to the closed-loop characteristics, e.g., stability? **Bandwidth**? Overshoot? Steady-state error?



Bandwidth Approximation

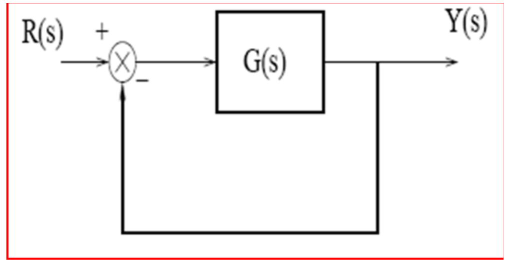
- **Approximation from open-loop TF**

  Assume a *second-order system*, the bandwidth frequency of the *closed-loop system* equals the frequency at which the *open-loop's* magnitude response is between -6 and - 7.5dB, assuming the open loop phase response is between -135 deg and -225 deg.

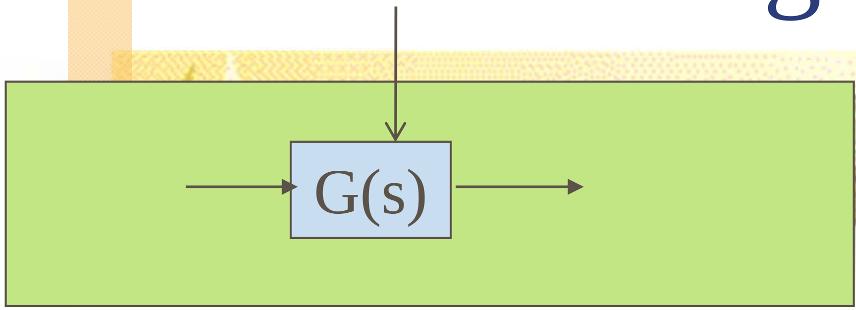
$$\omega_{BW} = \omega_n \sqrt{\left(1-2\zeta^2\right)+\sqrt{\zeta^4-4\zeta^2+2}} \qquad \omega_n = \frac{4}{T_s\zeta} \qquad \omega_{BW} = \frac{4}{T_s\zeta^2}\sqrt{\left(1-2\zeta^2\right)+\sqrt{\zeta^4-4\zeta^2+2}}$$

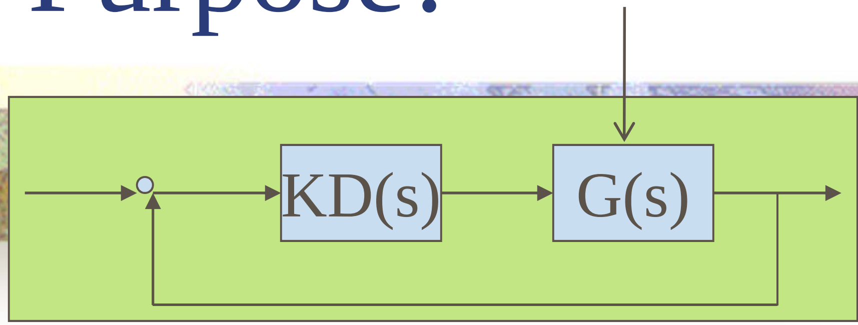
$$\omega_n = \frac{\pi}{T_p\sqrt{1-\zeta^2}} \qquad \omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{\left(1-2\zeta^2\right)+\sqrt{\zeta^4-4\zeta^2+2}}$$

- Relationship: **Wgc ≤ Wbw ≤ 2Wgc**



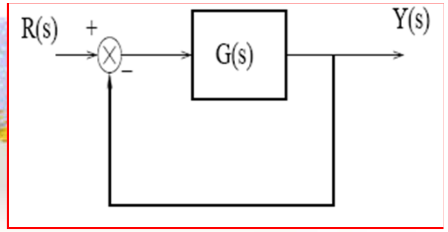
Can we further use Bode plot for Control Design Purpose?





For example,

Setup the relationship of the open-loop characteristics to the closed-loop characteristics, e.g., stability? Bandwidth? **Overshoot**? Steady-state error?

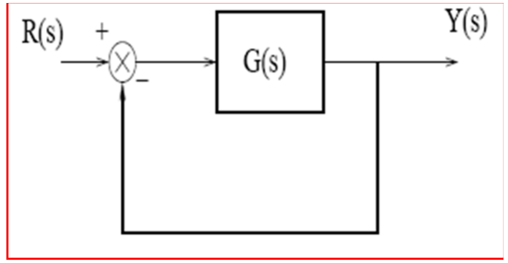


Damping Ratio Approximation

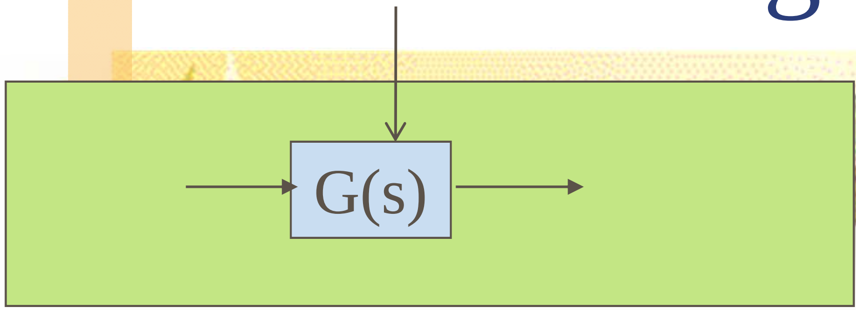
- For second-order systems, the closed-loop damping ratio is approximately equal to the phase margin divided by 100 if the phase margin is between 0 and 60 deg.

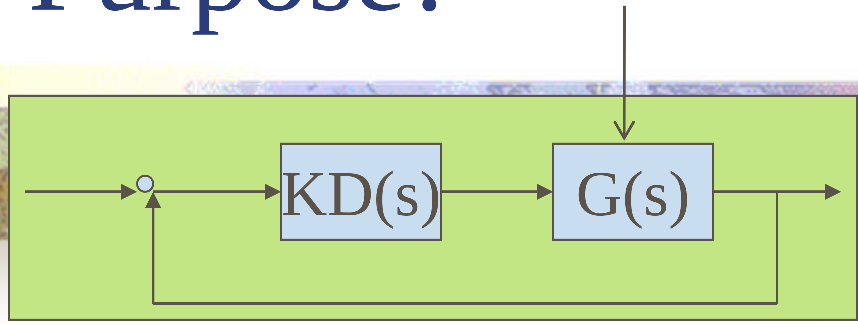
$$\xi \approx PM/100$$

- We can use this concept with caution if the phase margin is greater than 60 deg.



Can we further use Bode plot for Control Design Purpose?

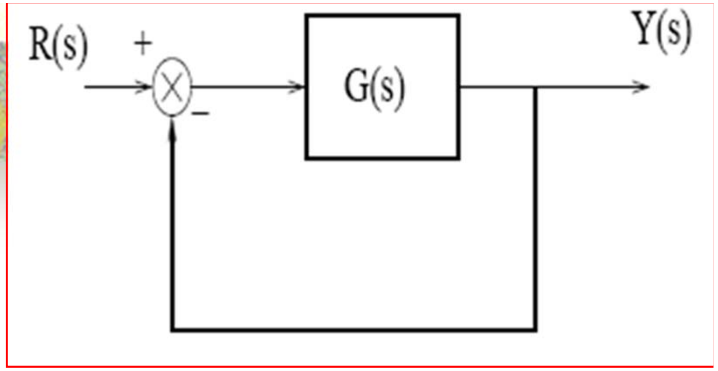




For example,

Setup the relationship of the open-loop characteristics to the closed-loop characteristics, e.g., stability? Bandwidth? Overshoot? **Steady-state error**?

Steady-State Errors



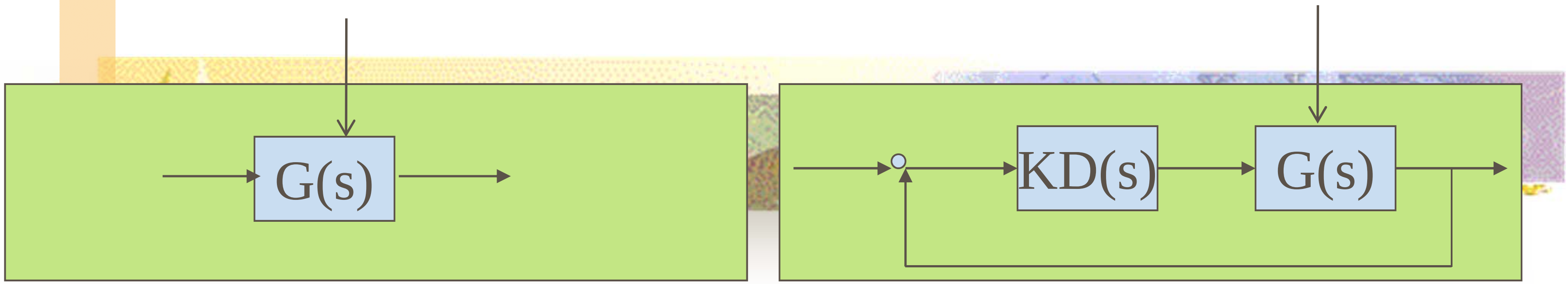
- The steady-state error of the closed-loop system will depend on the type of input (step, ramp, etc) as well as the (open-loop) system type (0, I, or II)

- Step Input (R(s) = 1/s): $e(\infty) = \dfrac{1}{1+\lim_{s\to 0} G(s)} = \dfrac{1}{1+K_p} \Rightarrow K_p = \lim_{s\to 0} G(s)$

- Ramp Input (R(s) = 1/s^2): $e(\infty) = \dfrac{1}{\lim_{s\to 0} sG(s)} = \dfrac{1}{K_v} \Rightarrow K_v = \lim_{s\to 0} sG(s)$

- Parabolic Input (R(s) = 1/s^3): $e(\infty) = \dfrac{1}{\lim_{s\to 0} s^2G(s)} = \dfrac{1}{K_a} \Rightarrow K_a = \lim_{s\to 0} s^2G(s)$

We indeed can use Bode plot for Control Design Purpose!



Setup the relationship of the open-loop characteristics to the closed-loop characteristics, e.g., **stability, Bandwidth, Overshoot, Steady-state errors**

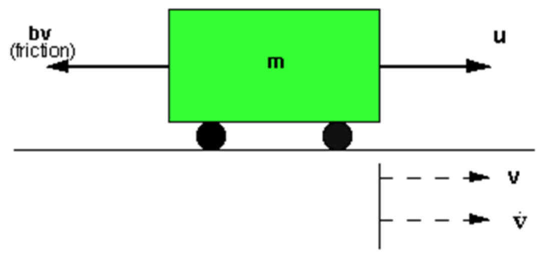
Remarks of Using Bode Plot

- **Precondition:** The system must be stable in open loop if we are going to design via Bode plots
- **Stability:** If the gain crossover frequency is less than the phase crossover frequency (i.e. Wgc < Wpc), then the closed-loop system will be stable
- **Damping Ratio:** For second-order systems, the closed-loop damping ratio is approximately equal to the phase margin divided by 100 if the phase margin is between 0 and 60 deg
- A very rough estimate that you can use is that the bandwidth is approximately equal to the **natural frequency**

http://www.engin.umich.edu/class/ctms/examples/cruise/cc.htm

Execise: Check Example:Cruise Control Problem

- **Considered system:** The inertia of the wheels is neglected, and it is assumed that friction is proportional to the car's speed, then the problem is reduced to the simple mass and damper system shown below.



- **System model:**

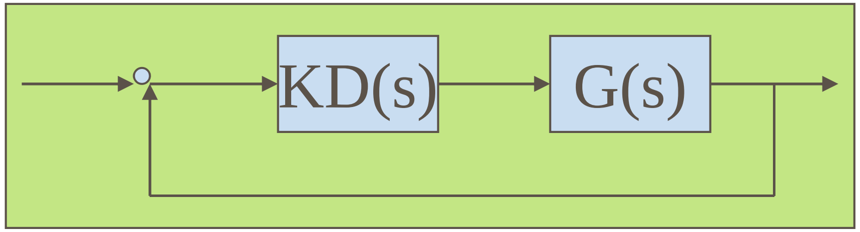
$$\frac{Y(s)}{U(s)} = \frac{1}{ms + b}$$

- **Design Specifications:** When the engine gives a 500 Newton force, the car will reach a maximum velocity of 10 m/s. An automobile should be able to accelerate up to that speed in less than 5 seconds. Since this is only a cruise control system, a 10% overshoot on the velocity will not do much damage. A 2% steady-state error is also acceptable for the same reason.

m=1000;
b=50; u=500;

Rise time < 5 sec
Overshoot < 10%
Steady state error < 2%

Design Example for next lecture....



- Plant model: G(s)=10/(1.25s+1)
- Requirement:
  - Zero steady state error for step input
  - Maximum overshoot must be less than 40%
  - Settling time must be less than 0.2 secs

Is it necessary to develop a controler? If so, how to develop what kind of controller?

Execise MM8

- Check slide page 15, 20 and 37