Digital Control

Lecture 1

Contents

Part I

Presentation

1 Discrete Transfer Functions

1.1 z-Transform

Definition of z-transform

For a discrete signal $e[k]$ with values $e_0, e_1, \ldots, e_k, \ldots$ the z-transform is given by:

z-Transform

$$
\begin{aligned}
E(z) &\widehat{=} \mathcal{Z}\{e[k]\} \\
&\widehat{=} \sum_{k=-\infty}^{\infty} e_k z^{-k}
\end{aligned}
$$

Discrete signal can be obtained by sampling a continuous signal with a sample time T

Description of a structured signal

Given a structural relationship between sequences a z-domain equivalent can be obtained:

Sequences

$$\sum_{k=-\infty}^{\infty} e_k z^{-k} = 1 + a_1 \sum_{k=-\infty}^{\infty} e_{k-1} z^{-k}$$

z-domain

$$E(z) = \frac{1}{1 - a_1 z^{-1}}$$

Blackboard example 1 (Obtaining structured expression):

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} e_k z^{-k} &= 1 + a_1 \sum_{k=-\infty}^{\infty} e_{k-1} z^{-k} \\
&= 1 + a_1 \sum_{j=-\infty}^{\infty} e_j z^{-(j+1)} \\
&= 1 + a_1 z^{-1} \sum_{j=-\infty}^{\infty} e_j z^{-j} \\
E(z) &= 1 + a_1 z^{-1} E(z) \\
&= \frac{1}{1 - a_1 z^{-1}}
\end{aligned}$$

1.2 Transfer Function

Discrete transfer functions

Given input signal $E(z)$ and output signal $U(z)$ a transfer function describing their relationship can be given as:

Transfer function

$$H(z) = \frac{U(z)}{E(z)}$$

Discrete transfer functions - continued

Transfer function expressions

$$\begin{aligned}
H(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \\
&= \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} \qquad n \geq m \\
&= \frac{b(z)}{a(z)}
\end{aligned}$$

Matlab

```
sys=tf(num,den,Ts)
```

Discrete System Models

Zeros and poles - General formula

$$H(z) = \frac{U(z)}{E(z)} = K\frac{\prod_{k=1}^{m}(z-z_k)}{\prod_{i=1}^{n}(z-p_i)}$$

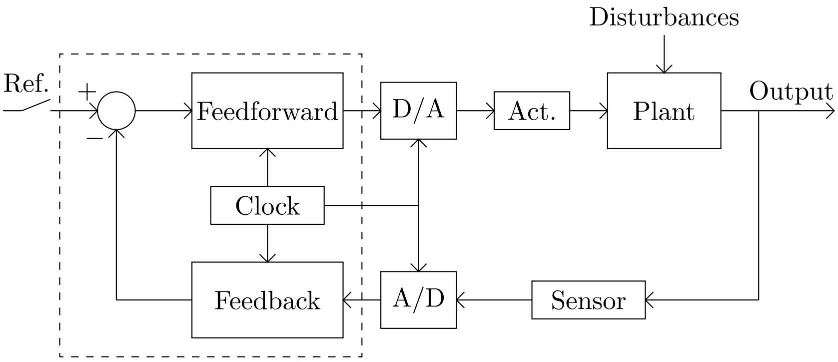
MATLAB

```
sys=zpk(z,p,k,Ts)
```

2 Discretization

2.1 Introducing Zero Order Hold

Basic Digital Control System



Zero Order Hold effect on continuous system

Basic property of ZOH

- Input to ZOH
  - Unit pulse, $\delta(kT)$
- Output from ZOH
  - Square pulse, $1(kT) - 1(kT - T)$

Effect on continuous system $G(s)$ after ZOH

$$\begin{aligned} Y(s) &= \frac{1}{s}G(s) - e^{-Ts}\frac{1}{s}G(s) \\ &= \left(1 - e^{-Ts}\right)\frac{G(s)}{s} \end{aligned}$$

Compensating for ZOH effect in z-transform

Incorporating ZOH in z-transform

$$
\begin{aligned}
G(z) &= \mathcal{Z}\left\{Y(kT)\right\} \\
&= \mathcal{Z}\left\{\left(1 - e^{-Ts}\right) \frac{G(s)}{s}\right\}
\end{aligned}
$$

Converting to z-domain

$$
G(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\}
$$

Blackboard example 2 (Converting ZOH effect to z-domain):

$$
\begin{aligned}
G(z) &= \mathcal{Z}\left\{\frac{G(s)}{s}\right\} - \mathcal{Z}\left\{e^{-Ts}\frac{G(s)}{s}\right\} \\
&= \mathcal{Z}\left\{\frac{G(s)}{s}\right\} - z^{-1}\mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\
&= \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\}
\end{aligned}
$$

ZOH discretization example

Continuous system

$$
G(s) = \frac{a}{s+a}
$$

ZOH equivalent

$$
G(z) = \frac{1 - e^{-aT}}{z - e^{-aT}}
$$

Blackboard example 3 (Discretization example):

$$
\begin{aligned}
G(s) &= \frac{a}{s+a} \\
G(z) &= \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\} \\
&= \left(1 - z^{-1}\right) \frac{z\left(1 - e^{-aT}\right)}{(z-1)\left(z - e^{-aT}\right)} \\
&= \frac{z-1}{z} \frac{z\left(1 - e^{-aT}\right)}{(z-1)\left(z - e^{-aT}\right)} \\
&= \frac{1 - e^{-aT}}{z - e^{-aT}}
\end{aligned}
$$

2.2 Numerical Integration

Numerical integration of differential equations

Continuous system

$$G(s) = \frac{U(s)}{E(s)} = \frac{a}{s+a}$$
$$\dot{u}(t) + au(t) = ae(t)$$

Solving differential equation

$$u(t) = u(kT - T) + \int_{kT-T}^{kT} -au(\tau) + ae(\tau)d\tau$$

Blackboard example 4 (Solving differential equation):

$$\begin{aligned} u(t) &= \int_0^t -au(\tau) + ae(\tau)d\tau \\ &= \int_0^{kT-T} -au(\tau) + ae(\tau)d\tau + \int_{kT-T}^{kT} -au(\tau) + ae(\tau)d\tau \\ &= u(kT - T) + \int_{kT-T}^{kT} -au(\tau) + ae(\tau)d\tau \end{aligned}$$

Blackboard example 5 (Discretization approximations):
Draw a continuous signal and plot approximations of forward rectangular rule, backward rectangular rule and trapezoidal rule.

Approximations of s for different rules

Forward rectangular rule

$$s \approx \frac{z-1}{T}$$

Backward rectangular rule

$$s \approx \frac{z-1}{Tz}$$

Trapezoidal rule

$$s \approx \frac{2}{T}\frac{z-1}{z+1}$$

Blackboard example 6:
Draw mappings of forward rule, backward rule, trapezoidal rule to z-domain.
Forward rule maps into area left of 1 in z-plane.
Backward rule maps into circle w. center $\frac{1}{2}$ and radius $\frac{1}{2}$ in z-plane.
Trapezoidal rule maps into unit circle of z-plane.

2.3 Zero-Pole Matching

Direct mapping of zeros/poles

Pole mapping

$$s_p = -a + jb \qquad \Rightarrow z_p = e^{-aT}\angle bT$$

Zero mapping (finite)

$$s_z = -a + jb \qquad \Rightarrow z_z = e^{-aT}\angle bT$$

Direct mapping of zeros/poles - continued

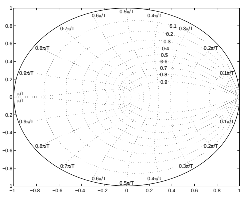
Zero mapping (infinite, no delay)

$$s_z = \infty \qquad \Rightarrow z_z = -1$$

Zero mapping (infinite, with n sample delay)

$$s_z = \infty \qquad \Rightarrow z_z = \infty \text{ (n zeros)}$$
$$s_z = \infty \qquad \Rightarrow z_z = -1 \text{ (remaining zeros)}$$

The Discrete z-plane



Direct mapping of zeros/poles - continued

Gain matching

$$H(s)|_{s=\omega_0} = H(z)|_{z=e^{T\omega_0}}$$

ω_0 is usually 0 in order to match steady-state gains

Discretization in Matlab

Matlab

`sysd=c2d(sys,Ts,method)`

method:

- 'zoh': Zero order hold
- 'foh': First order hold (academic)
- 'tustin': Bilinear approximation (trapezoidal)
- 'prewarp': Tustin with a specific frequency used for prewarp
- 'matched': Matching continuous poles with discrete

2.4 Stability

Stability - Impulse Response

Continuous systems

The system is BIBO stable if and only if the impulse response $h(t)$ is absolutely integrable

Discrete systems

The system is BIBO stable if and only if the impulse response $h[n]$ is absolutely summable

Stability - Characteristic Roots

Asymptotic internal stability

Continuous systems

All poles of the system are strictly in the LHP of the s-plane

Discrete systems

All poles of the system are strictly inside the unit circle of the z-plane

poles are the roots of the characteristic equation

Exercises

Book: Digital Control

- Problem 6.3 a.i-a.vi+b (use MATLAB when possible)
- Problem 6.4 a.i-a.vi+b (use MATLAB when possible)