Digital Control
Lecture 1
Questions

• How to represent digital signals & systems?
• How to obtain an equivalent digital representation of a analog system?
• How to analyze digital signal/system features?
• How to develop a digital controller?
• What needs to be concerned when implementing a digital controller?
Outline

1. Discrete Transfer Functions
2. Discretization
Outline

1. Discrete Transfer Functions
   - $z$-Transform
   - Transfer Function

2. Discretization
   - Introducing Zero Order Hold
   - Numerical Integration
   - Zero-Pole Matching
   - Stability
Definition of \( z \)-transform

For a discrete signal \( e[k] \) with values \( e_0, e_1, \ldots, e_k, \ldots \) the \( z \)-transform is given by:

\[
E(z) \triangleq \mathcal{Z}\{e[k]\} = \sum_{k=-\infty}^{\infty} e_k z^{-k}
\]

Discrete signal can be obtained by sampling a continuous signal with a sample time \( T \)
Description of a structured signal

Given a structural relationship between sequences a z-domain equivalent can be obtained:

\[
\sum_{k=-\infty}^{\infty} e_k z^{-k} = 1 + a_1 \sum_{k=-\infty}^{\infty} e_{k-1} z^{-k}
\]

### Sequences

\[
E(z) = \frac{1}{1 - a_1 z^{-1}}
\]
Discrete transfer functions

Given input signal $E(z)$ and output signal $U(z)$ a transfer function describing their relationship can be given as:

$$H(z) = \frac{U(z)}{E(z)}$$
Transfer function expressions

\[ H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \]
\[ = \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} \quad n \geq m \]
\[ = \frac{b(z)}{a(z)} \]

**MATLAB**

\[ \text{sys} = \text{tf(num,den,Ts)} \]
Discrete System Models

Zeros and poles - General formula

\[ H(z) = \frac{U(z)}{E(z)} = K \frac{\prod_{k=1}^{m}(z - z_k)}{\prod_{i=1}^{n}(z - p_i)} \]

MATLAB

sys=zpk(z,p,k,Ts)
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Basic Digital Control System

- Ref.
- Feedforward
- Clock
- Feedback
- D/A
- Act.
- Plant
- A/D
- Sensor
- Disturbances
- Output
Zero Order Hold effect on continuous system

Basic property of ZOH
- Input to ZOH
  - Unit pulse, \( \delta(kT) \)
- Output from ZOH
  - Square pulse, \( 1(kT) - 1(kT - T) \)

Effect on continuous system \( G(s) \) after ZOH

\[
Y(s) = \frac{1}{s} G(s) - e^{-Ts} \frac{1}{s} G(s)
\]

\[
= \left(1 - e^{-Ts}\right) \frac{G(s)}{s}
\]
Compensating for ZOH effect in $z$-transform

Incorporating ZOH in $z$-transform

\[ G(z) = Z \{ Y(kT) \} \]
\[ = Z \left\{ \left(1 - e^{-Ts} \right) \frac{G(s)}{s} \right\} \]

Converting to $z$-domain

\[ G(z) = (1 - z^{-1}) Z \left\{ \frac{G(s)}{s} \right\} \]
**ZOH discretization example**

**Continuous system**

\[
G(s) = \frac{a}{s + a}
\]

**ZOH equivalent**

\[
G(z) = \frac{1 - e^{-aT}}{z - e^{-aT}}
\]
Numerical integration of differential equations

Continuous system

\[ G(s) = \frac{U(s)}{E(s)} = \frac{a}{s + a} \]
\[ \dot{u}(t) + au(t) = ae(t) \]

Solving differential equation

\[ u(t) = u(kT - T) + \int_{kT-T}^{kT} -au(\tau) + ae(\tau) d\tau \]
Approximations of $s$ for different rules

**Forward rectangular rule**

$$s \approx \frac{z - 1}{T}$$

**Backward rectangular rule**

$$s \approx \frac{z - 1}{Tz}$$

**Trapezoidal rule**

$$s \approx \frac{2(z - 1)}{T(z + 1)}$$
Direct mapping of zeros/poles

Pole mapping

\[ s_p = -a + jb \quad \Rightarrow z_p = e^{-aT} \angle bT \]

Zero mapping (finite)

\[ s_z = -a + jb \quad \Rightarrow z_z = e^{-aT} \angle bT \]
Direct mapping of zeros/ poles - continued

Zero mapping (infinite, no delay)

\[ s_z = \infty \Rightarrow z_z = -1 \]

Zero mapping (infinite, with \( n \) sample delay)

\[ s_z = \infty \Rightarrow z_z = \infty \ (n \text{ zeros}) \]
\[ s_z = \infty \Rightarrow z_z = -1 \ (\text{remaining zeros}) \]
Gain matching

\[ H(s)|_{s=\omega_0} = H(z)|_{z=e^{T\omega_0}} \]

\( \omega_0 \) is usually 0 in order to match steady-state gains
Discretization in MATLAB

```matlab
sysd = c2d(sys, Ts, method)
```

**Method:**
- `'zoh'`: Zero order hold
- `'foh'`: First order hold (academic)
- `'tustin'`: Bilinear approximation (trapezoidal)
- `'prewarp'`: Tustin with a specific frequency used for prewarp
- `'matched'`: Matching continuous poles with discrete
Continuous systems

The system is BIBO stable if and only if the impulse response $h(t)$ is absolutely integrable.

Discrete systems

The system is BIBO stable if and only if the impulse response $h[n]$ is absolutely summable.
Stability - Characteristic Roots

Asymptotic internal stability

Continuous systems
All poles of the system are strictly in the LHP of the $s$-plane

Discrete systems
All poles of the system are strictly inside the unit circle of the $z$-plane
Book: Digital Control

- Problem 6.3 a.i-a.vi+b (use MATLAB when possible)
- Problem 6.4 a.i-a.vi+b (use MATLAB when possible)