Digital Control

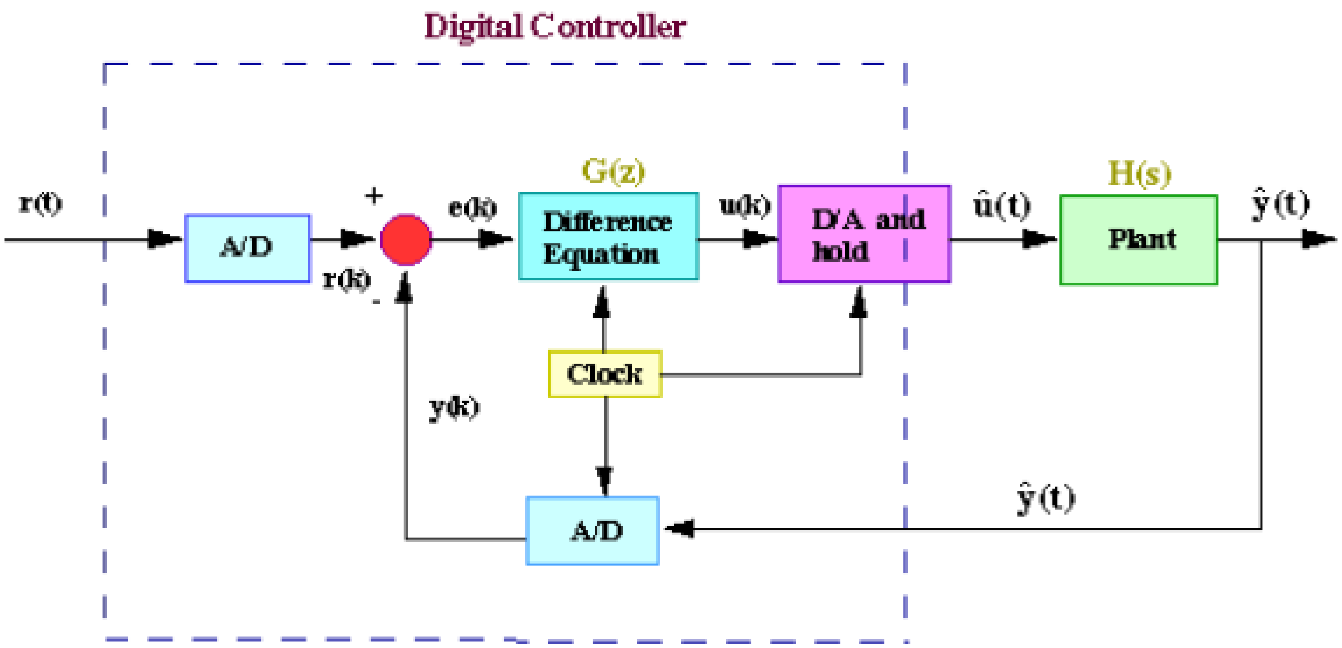
Lecture 2

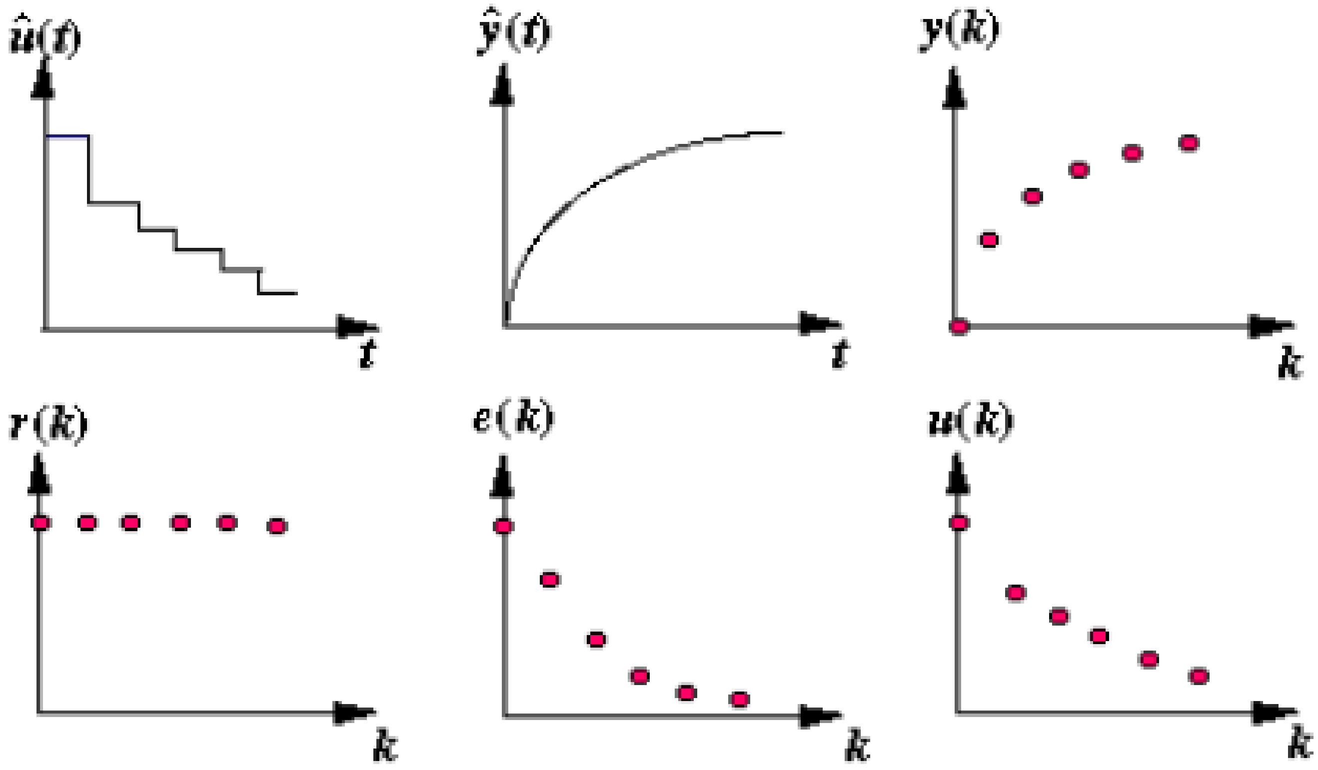
Outline

1. Sampling Rate Selection
2. Equivalents Between Continuous and Digital Systems

What have we talked about in MM1?

- Discrete transfer function
- Discretization (ZOH)


Digital Controller
r(t)
A/D
+
r(k)
-
e(k)
G(z)
Difference Equation
u(k)
D/A and hold
û(t)
H(s)
Plant
ŷ(t)
Clock
y(k)
A/D
ŷ(t)


û(t)
t
ŷ(t)
t
y(k)
k
r(k)
k
e(k)
k
u(k)
k

Discrete transfer functions - continued

Transfer function expressions

$$
\begin{aligned}
H(z) &= \frac{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}} \\
&= \frac{b_0 z^n + b_1 z^{n-1} + \cdots + b_m z^{n-m}}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \cdots + a_n} \qquad n \geq m \\
&= \frac{b(z)}{a(z)}
\end{aligned}
$$

Matlab

```
sys=tf(num,den,Ts)
```

Zero Order Hold effect on continuous system

Basic property of ZOH

- Input to ZOH
  - Unit pulse, $\delta(kT)$
- Output from ZOH
  - Square pulse, $1(kT) - 1(kT - T)$

Converting to z-domain

$$G(z) = \left(1 - z^{-1}\right) \mathcal{Z}\left\{\frac{G(s)}{s}\right\}$$

Discretization in MATLAB

MATLAB

```
sysd=c2d(sys,Ts,method)
```

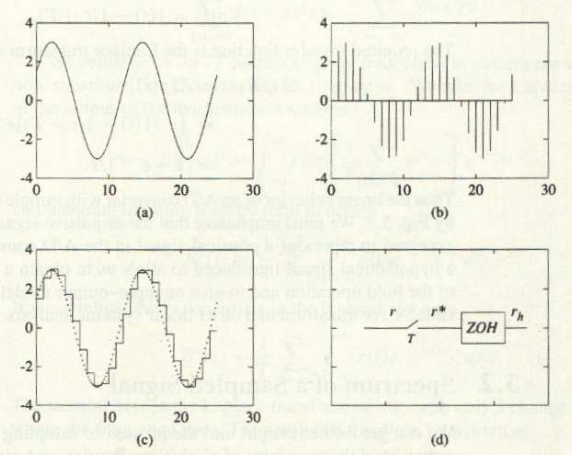
method:

- 'zoh': Zero order hold
- 'foh': First order hold (academic)
- 'tustin': Bilinear approximation (trapezoidal)
- 'prewarp': Tustin with a specific frequency used for prewarp
- 'matched': Matching continuous poles with discrete

Outline

1. Sampling Rate Selection
   - Sampling Theorem
   - Smoothness
   - Effect of Noise
2. Equivalents Between Continuous and Digital Systems
   - Sampling Effect
   - System Specifications

Sampling of a continuous signal



Representations of sampled signals - time domain

Sum of impulses

$$r^*(t) = \sum_{k=-\infty}^{\infty} r(t)\delta(t-kT)$$

Fourier series

$$r^*(t) = r(t)\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn2\pi t/T}$$

Representations of sampled signals - Laplace domain

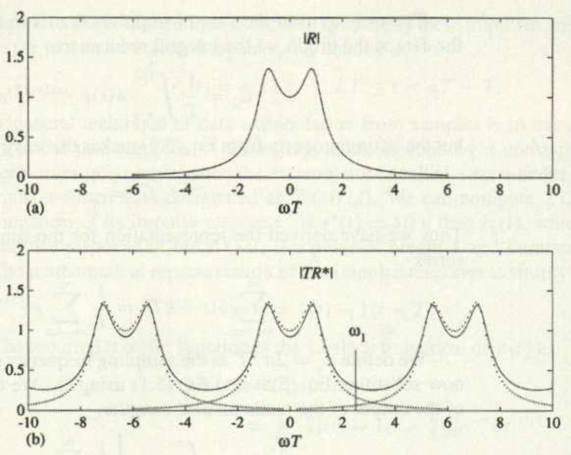
Laplace representation

$$R^*(s) = \int_{-\infty}^{\infty} r(t)\left\{\frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jnt\omega_s}\right\} e^{-st}dt \qquad \omega_s = \frac{2\pi}{T}$$

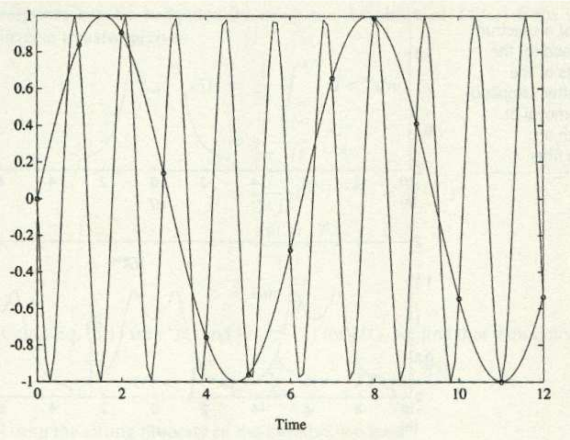
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} r(t)e^{-(s-jn\omega_s)t}dt$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} R(s-jn\omega_s)$$

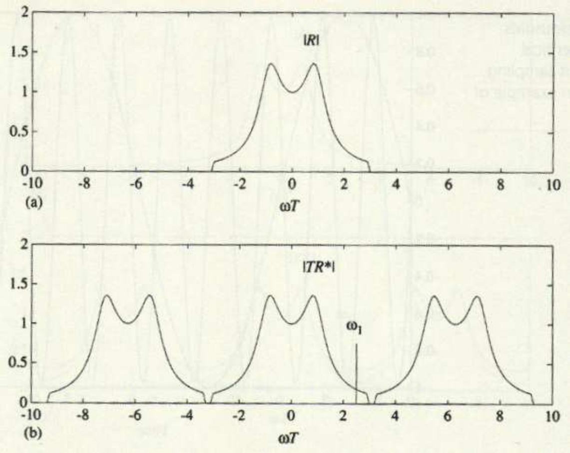
Spectrum of sampled signal



Aliasing



Anti-aliasing



Use anti-aliasing filter before sampling a signal!

Sampling theorem

Theoretical lower limit

$$\frac{\omega_s}{\omega_b} > 2$$

ω_s: sampling frequency

ω_b: required closed-loop bandwidth

Not practical!

Smooth response

Practical limits

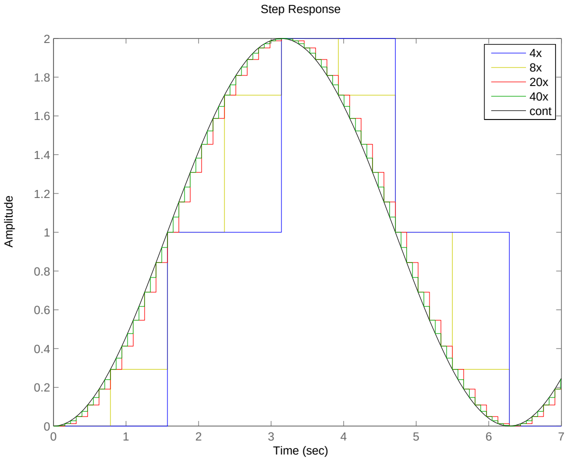
$$20 < \frac{\omega_s}{\omega_b} < 40$$

ω_s: sampling frequency
ω_b: required closed-loop bandwidth

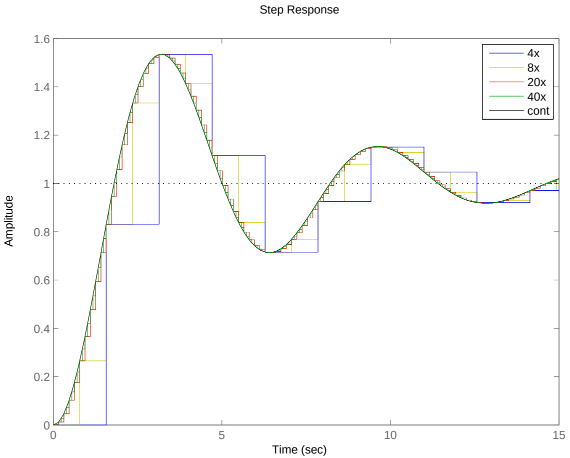
External requirements to minimizing sampling time

- Bandwidth of reference
- Human interaction
- Sensitivity to delays

Sampling of sinusoid

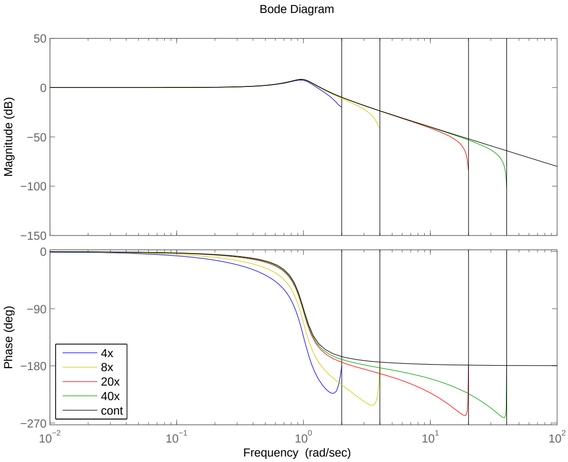


Sampling of damped system

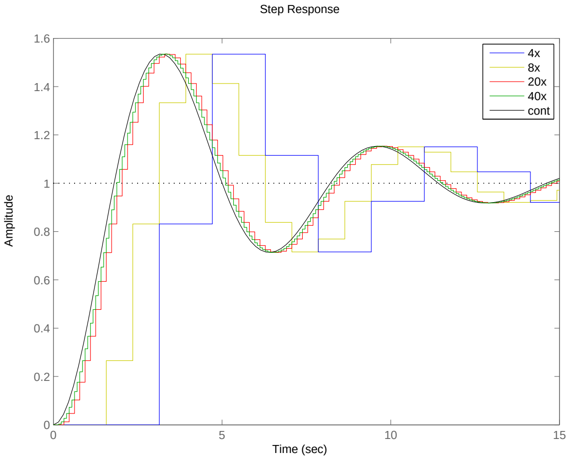


Rule-of-thumb: 6-10 samples in closed-loop rise-time

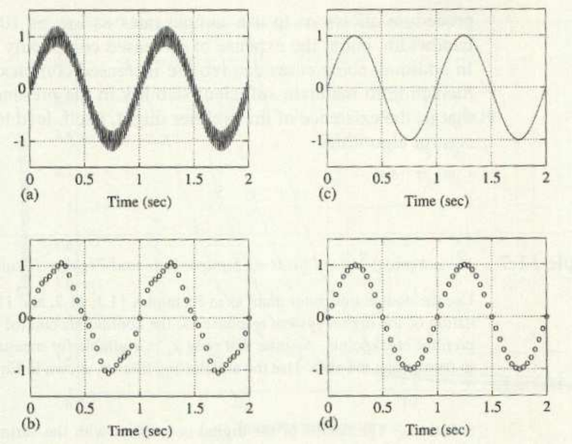
Effect of sampling on frequency response



Sampling of damped system with 1 sample delay



Effect of noise on sampling



Prefiltering is important!

Requirements to prefilter

Breakpoint of prefilter, ω_c

$$\omega_b < \omega_c < \frac{\omega_s}{2}$$

ω_s: sampling frequency
ω_b: required closed-loop bandwidth
ω_c: breakpoint of filter

Prefilter should filter lowest noise frequencies while not disturbing highest system frequencies!

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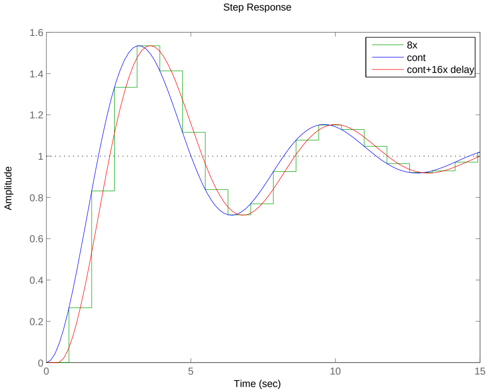
Reconstruction of signal from ZOH



Reconstructed signal corresponds to continuous system with half-sample delay

Accommodating for sampling effect delay

Approximation of delay

$$e^{-sT/2} \approx \frac{2/T}{s + 2/T}$$

Accommodation

Investigate effect of half-sample delay on continuous system before digitizing:

- Root locus wrt. T
- Analyze using frequency based methods
  - Effect on phase margin/damping

Digital equivalents of continuous-time specifications

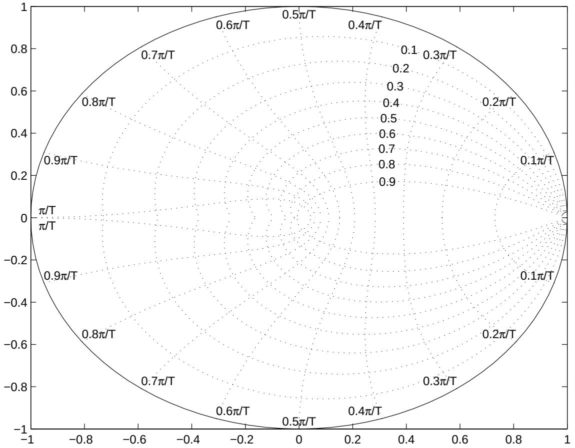
Transient response

- Overshoot/damping, M_p, ζ
- Rise time, ω_n
- Settling time, ω_n, ζ, σ

Steady-state response

- Steady-state: $z \longrightarrow 1$, $(\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)F(z))$
- System type: number of pure integrators in open-loop, $(z = 1)$
- System input
  - Step, $\frac{z}{z-1}$
  - Ramp, $\frac{Tz}{(z-1)^2}$
  - Parabola, $\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}$

Time-domain specifications in z-domain



Digital Control Tutorial

[Introduction](#)
[Zero-order hold equivalence](#)
[Conversion using c2dm](#)
[Stability and transient response](#)
[Discrete Root-Locus](#)

Key Matlab Commands used in this tutorial are: c2dm pzmap zgrid dstep stairs rlocus

Note: Matlab commands from the control system toolbox are highlighted in red.

For analyzing the transient response from pole locations in the z-plane, the following three equations used in continuous system designs are still applicable.

$$\xi\omega_n \geq \frac{4.6}{Ts}$$

$$\omega_n \geq \frac{1.8}{Tr}$$

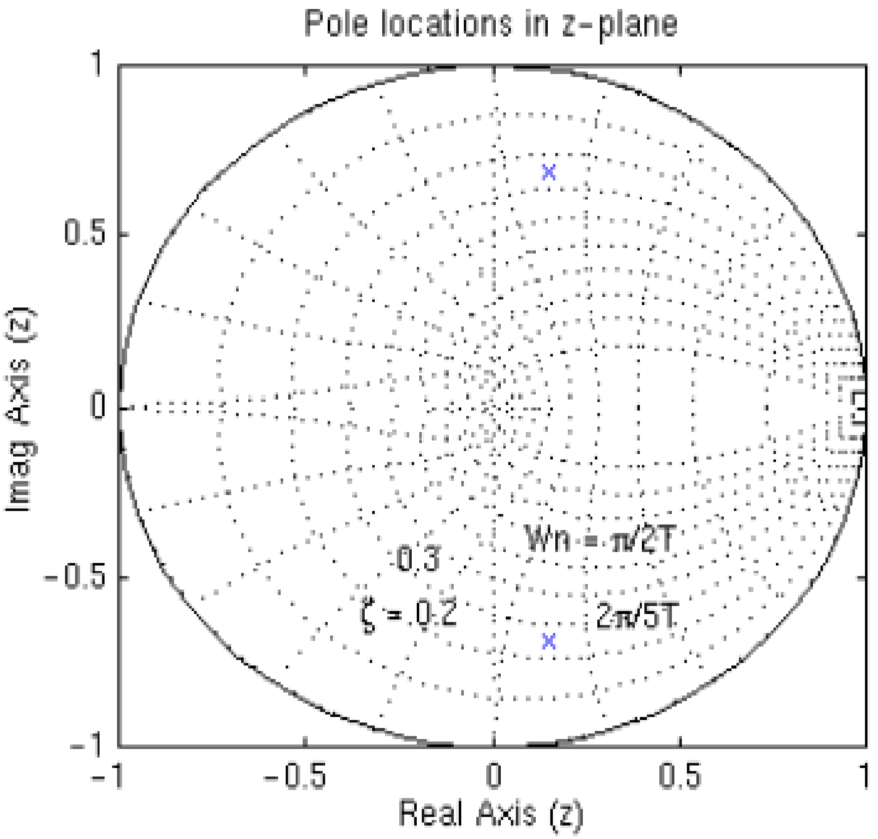
$$\xi \geq \sqrt{\frac{\left(\ln Mp/\pi\right)^2}{1+\left(\ln Mp/\pi\right)^2}}$$

where

- zeta = Damping ratio
- Wn = Natural frequency **(rad/sec)**
- Ts = Settling time
- Tr = Rise time
- Mp = Maximum overshoot

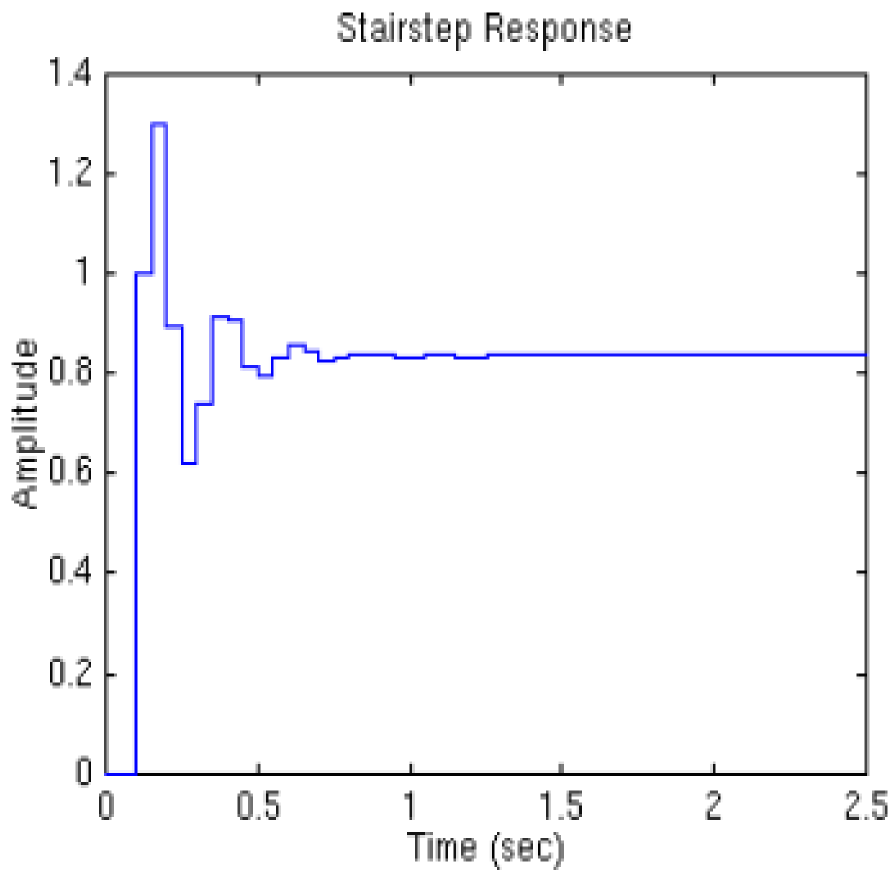
Important:

The natural frequency (Wn) in z-plane has the unit of rad/sample, but when you use the equations shown above, the Wn must be in the unit of rad/sec.



From this plot, we see poles are located approximately at the natural frequency of 9pi/20T (rad/sample) and the damping ratio of 0.25. Assuming that we have the sampling time of 1/20 sec (which leads to Wn = 28.2 **rad/sec**) and using three equations shown above, we can determine that this system should have the rise time of 0.06 sec, the settling time of 0.65 sec and the maximum overshoot of 45% (0.45 more than the steady-state value). Let's obtain the step response and see if these are correct. Add the following commands to the above m-file and rerun it in the command window. You should get the following step response.

```
[x] = dstep (numDz,denDz,51);
t = 0:0.05:2.5;
stairs (t,x)
```



As you can see from the plot, all of the rise time, the settling time and the overshoot came out to be what we expected. We proved you here that we can use the locations of poles and the above three equations to analyze the transient response of the system.