Digital Control
Lecture 3
1 Digital Controller Design

2 Lead Compensator for Antenna - Design Example
What have we talked about in MM2?

- Sampling rate selection
- Equivalents between continuous & digital Systems
### Sampling theorem

**Theoretical lower limit**

\[
\frac{\omega_s}{\omega_b} > 2
\]

- \(\omega_s\): sampling frequency
- \(\omega_b\): required closed-loop bandwidth

### Smooth response

**Practical limits**

\[
20 < \frac{\omega_s}{\omega_b} < 40
\]

- \(\omega_s\): sampling frequency
- \(\omega_b\): required closed-loop bandwidth
Requirements to prefilter

**Breakpoint of prefilter, \( \omega_c \)**

\[
\omega_b < \omega_c < \frac{\omega_s}{2}
\]

- \( \omega_s \): sampling frequency
- \( \omega_b \): required closed-loop bandwidth
- \( \omega_c \): breakpoint of filter

**Prefilter should filter lowest noise frequencies while not disturbing highest system frequencies!**
Accommodating for sampling effect delay

Approximation of delay

\[ e^{-sT/2} \approx \frac{2/T}{s + 2/T} \]

Accommodation

Investigate effect of half-sample delay on continuous system before digitizing:

- Root locus wrt. \( T \)
- Analyze using frequency based methods
  - Effect on phase margin/damping
Digital equivalents of continuous-time specifications

**Transient response**
- Overshoot/damping, $M_p, \zeta$
- Rise time, $\omega_n$
- Settling time, $\omega_n, \zeta, \sigma$

**Steady-state response**
- Steady-state: $z \to 1$, $(\lim_{k \to \infty} f(k) = \lim_{z \to 1} (z - 1)F(z))$
- System type: number of pure integrators in open-loop, $(z = 1)$
- System input
  - Step, $\frac{z}{z-1}$
  - Ramp, $\frac{Tz}{(z-1)^2}$
  - Parabola, $\frac{T^2}{2} \frac{z(z+1)}{(z-1)^3}$
Outline

1. Digital Controller Design
   - Emulation Method for Digital Control

2. Lead Compensator for Antenna - Design Example
   - Effect of Sample Times
   - Accommodation for Sampling Delay
   - Effect of Sampling Method
Digital controller can be obtained using:

- Emulation, which finds the discrete equivalent of a continuous controller
- Direct discrete design (next lecture)
Frequency Issues

Continuous Systems
For a minimum-phase transfer function, the phase is uniquely determined by the magnitude curve:

$$\angle G(j\omega) \approx n \times 90^\circ$$

where $n$ is the slope of $G(j\omega)$ in units of decade of amplitude

Discrete Systems
The amplitude and phase relationship is lost!
The prediction of stability from the amplitude curve alone for minimum-phase systems is lost
It is typically necessary to determine both magnitude and phase for discrete systems
Emulation Method

1. A continuous controller is designed
2. Sample time is selected
3. Discrete equivalent is computed
4. Evaluation of design
Outline

1. Digital Controller Design
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2. Lead Compensator for Antenna - Design Example
   - Effect of Sample Times
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   - Effect of Sampling Method
Case Study: Antenna Control

General System Model:

\[ J\ddot{\theta} + B\dot{\theta} = T_c + T_d \]

Discarding the disturbances \( T_d \) gives the transfer function:

\[
\frac{\Theta(s)}{U(s)} = \frac{1}{s \left( \frac{s}{a} + 1 \right)}
\]

where \( a = \frac{B}{J} = 0.1 \) and \( u(t) = \frac{T_c(t)}{B} \).

Design Specifications:

- Overshoot to a step input less than 16% (PM \( \approx 55 \))
- Settling time to 1% in less than 10s
- Tracking error to ramp of slope \( 0.01 \frac{\text{rad}}{\text{sec}} \) less than 0.01 rad
- Sampling time to give at least 10 samples in a rise-time
Step 1
Design the low frequency gain $K$ with respect to the steady-state error specification
Antenna system case: $K = 1$

Step 2
Determine the needed phase lead

```matlab
sys=tf(1,[10 1 0]);
margin(sys)
```

PM=18 at $\omega = 0.308$
Lead Compensator Design for Antenna (FC pp. 375)

Step 3
Using lead contribution of $\phi_{max} = 45$ should result in PM=63 which is 8 more than needed.

Step 4
Determine:

$$\alpha = \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}} = \frac{1 - \sin 45}{1 + \sin 45} = 0.1716$$

Step 5

$$T = \frac{1}{\omega_{max} \sqrt{\alpha}} = \frac{1}{\frac{\omega_n}{2} \sqrt{\alpha}} = \frac{2}{0.92 \sqrt{\alpha}} = 5.248$$

Giving a zero in $s = -\frac{1}{T} = -0.19$ and a pole in $s = -\frac{1}{\alpha T} = -1.11$. 
Step 6

Draw the compensated frequency response, check PM
Using the formulation:

\[ D(s) = \frac{Ts + 1}{\alpha Ts + 1} \]

we use:

sysD=tf([5.3 1],[0.9 1])
sysC=sys*sysD
margin(sysC)
step(feedback(sysC,1))
Lead Compensator Design for Antenna (FC pp. 375)

- **Bode Diagram**
  - $G_m = \infty$ dB (at Inf rad/sec)
  - $P_m = 56.3$ deg (at 0.505 rad/sec)

- **Step Response**
  - System: untitled1
  - Time (sec): 5.65
  - Amplitude: 1.16
  - System: untitled1
  - Time (sec): 13.3
  - Amplitude: 1.01

**Figure:** Frequency response

**Figure:** Step response
Step 7

**Step 7**: Iterate on the design until all specifications are met

```matlab
sysD=tf([10 1],[1 1])
sysC=sys*sysD
margin(sysC)
sysCL=feedback(sysC,1)
step(sysCL)
```
Lead Compensator Design for Antenna (FC pp. 375)

**Figure:** Frequency response

**Figure:** Step response
Digital Lead Compensator for Antenna - Fast Sampling

Continuous lead controller

\[ D(s) = \frac{10s + 1}{s + 1} \]

Digitization - Fast Sample Rate

```matlab
sysc=tf(1,[10 1 0]);
lead=tf([10 1],[1 1]);
syslead=sysc*lead;
Ts=1/20;
leadd1=c2d(lead,Ts,'zoh');
sysd=c2d(sysc,Ts,'zoh');
syscld=feedback(sysd*leadd1,1);
step(syscld)
```
Digital Lead Compensator for Antenna - Fast Sampling

Step Response

System: syscldfast
Time (sec): 3.65
Amplitude: 1.18

System: syscldfast
Time (sec): 8.75
Amplitude: 0.99

Lecture 3  Digital Control
**Continuous lead controller**

\[
D(s) = \frac{10s + 1}{s + 1}
\]

**Digitization - Slow Sample Rate**

```matlab
sysc=tf(1,[10 1 0]);
lead=tf([10 1],[1 1]);
syslead=sysc*lead;
Ts=1/2;
leadd1=c2d(lead,Ts,'zoh');
sysd=c2d(sysc,Ts,'zoh');
syscld=feedback(sysd*leadd1,1);
step(syscld)
```
Digital Lead Compensator for Antenna - Slow Sampling

Step Response

- System: syscldslow
  - Time (sec): 3.35
  - Amplitude: 1.35

- System: syscldslow
  - Time (sec): 13.5
  - Amplitude: 0.99

Lecture 3  Digital Control
Effect of Sample Time on Step Response

![Step Response Graph]

- Continuous
- Fast sampling
- Slow sampling

Time (sec) vs. Amplitude graph showing the step response for different sample times.
Effect of Sample Time on Frequency Response

Bode Diagram

- Magnitude (dB)
- Phase (deg)
- Frequency (rad/sec)

Lecture 3   Digital Control
Effect of Sample Time on Pole Locations

Pole–Zero Map

- System: syscldslow
  - Pole: 0.737 + 0.422i
  - Damping: 0.3
  - Overshoot (%): 37.3
  - Frequency (rad/sec): 1

- System: syscldfast
  - Pole: 0.975 − 0.0433i
  - Damping: 0.482
  - Overshoot (%): 17.7
  - Frequency (rad/sec): 1.01

- System: untitled1
  - Pole: −0.5 + 0.866i
  - Damping: 0.5
  - Overshoot (%): 16.3
  - Frequency (rad/sec): 1
Incorporating Sampling Delay in System

**Continuous System**

\[ G(s) = \frac{1}{s(10s + 1)} \]

**Continuous System with Delay**

\[ G_d(s) = \frac{2/T}{s + 2/T} \frac{1}{s(10s + 1)} \]
Lead Compensator for System using Slow Sampling Rate

Inserting $T = 1/2$

$$G_d(s) = \frac{2/T}{s + 2/T} \frac{1}{s(10s + 1)}$$

$$= \frac{4}{s(s + 4)(10s + 1)}$$
## Lead Compensator Design for Antenna

### Step 1

Design the low frequency gain $K$ with respect to the steady-state error specification.

Steady-state unchanged from original system: $K = 1$

### Step 2

Determine the needed phase lead.

```matlab
sys = tf(1,[10 41 4 0]);
margin(sys)
```

PM = 14 at $\omega = 0.308$
Lead Compensator Design for Antenna

**Step 3**
Using lead contribution of $\phi_{max} = 50$ should result in $PM=64$ which is 9 more than needed.

**Step 4**
Determine:

$$\alpha = \frac{1 - \sin \phi_{max}}{1 + \sin \phi_{max}} = \frac{1 - \sin 50}{1 + \sin 50} = 0.1325$$

**Step 5**

$$T = \frac{1}{\omega_{max} \sqrt{\alpha}} = \frac{1}{0.4 \sqrt{(0.1325)}} = 6.869$$

Giving a zero in $s = -\frac{1}{T} = -0.1456$ and a pole in $s = -\frac{1}{\alpha T} = -1.099$. 
Lead Compensator Design for Antenna

Step 6

Draw the compensated frequency response, check PM
Using the formulation:

\[ D(s) = \frac{Ts + 1}{\alpha Ts + 1} \]

we use:

```matlab
sysD = tf([6.9 1], [0.9 1])
sysC = sys * sysD
margin(sysC)
step(feedback(sysC, 1))
```
Lead Compensator Design for Antenna

**Bode Diagram**

$G_m = 16.9 \, \text{dB (at 2.05 rad/sec)}$, $P_m = 48.6 \, \text{deg (at 0.607 rad/sec)}$

**Figure:** Frequency response

**Step Response**

System: untitled1
- Time (sec): 4.76
- Amplitude: 1.21

System: untitled1
- Time (sec): 13.6
- Amplitude: 1.01

**Figure:** Step response
Step 7: Iterate on the design until all specifications are met

```matlab
sysD=tf([7.5 1],[0.68 1])
sysC=sys*sysD
margin(sysC)
sysCL=feedback(sysC,1)
step(sysCL)
```
Lead Compensator Design for Antenna

Bode Diagram

- Frequency response

Step Response

- Step response

Figure: Frequency response

Figure: Step response
Continuous lead controller

\[ D(s) = \frac{7.5s + 1}{0.68s + 1} \]

Digitization - Slow Sample Rate

```matlab
sysc=tf(1,[10 1 0]);
lead=tf([7.5 1],[0.68 1]);
syslead=sysc*lead;
Ts=1/2;
leadd1=c2d(lead,Ts,'zoh');
sysd=c2d(sysc,Ts,'zoh');
syscld=feedback(sysd*leadd1,1);
step(syscld)
```
Digital Lead Compensator for Antenna - Comparison

Figure: Frequency response

Figure: Step response
Discretization in **MATLAB**

MATLAB

```matlab
sysd=c2d(sys,Ts,method)
```

*method:*
- `'zoh'`: Zero order hold
- `'foh'`: First order hold (academic)
- `'tustin'`: Bilinear approximation (trapezoidal)
- `'prewarp'`: Tustin with a specific frequency used for prewarp
- `'matched'`: Matching continuous poles with discrete
Digital Controller Design
Lead Compensator for Antenna - Design Example
Exercises

Discretization of Lead Compensator - Fast Sample Rate

Figure: Frequency response

Figure: Step response
Discretization of Lead Compensator - Fast Sample Rate

Pole–Zero Map

- ZOH
- Tustin
- Matched
Discretization of Lead Compensator - Slow Sample Rate

**Figure:** Frequency response

**Figure:** Step response
Discretization of Lead Compensator - Slow Sample Rate

Pole–Zero Map

Real Axis

Imaginary Axis

-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1

-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1

0.1π/T 0.2π/T 0.3π/T 0.4π/T 0.5π/T 0.6π/T 0.7π/T 0.8π/T 0.9π/T π/T

ZOH Tustin Matched
Some important things to remember

**Discretization of compensator**

Use the method suited for implementation in the system

**Discrete equivalent of plant**

- Use method corresponding to implementation (usually ZOH)
- Simulink can combine discrete compensator with continuous plant (digitization of plant not necessary)
Book: Digital Control

- Problem 7.4
- Problem 7.5
- Problem 7.7