5 Kalman filters

5.1 Scalar Kalman filter

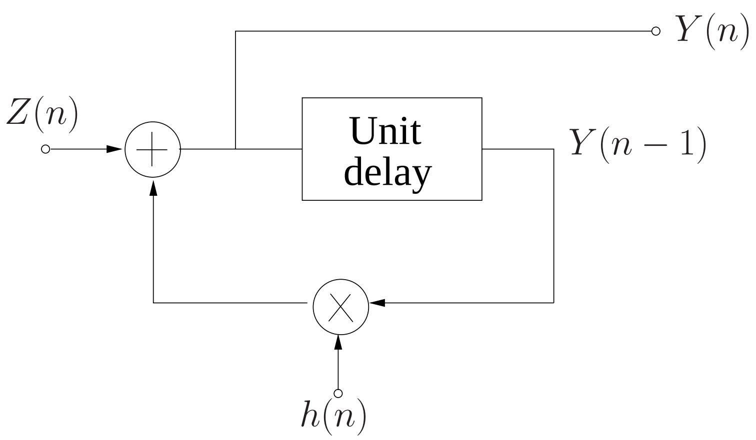
5.1.1 Signal model

- **System model**

  $\{Y(n)\}$ is an unobservable sequence which is described by the following ***state or system equation***:

$$Y(n) = h(n)Y(n-1) + Z(n), n = 1, 2, \ldots \tag{5.1}$$

Block Diagram Representation of (5.1)



Initialization:

$Y(0)$ is a random variable whose expectation $\mu_{Y(0)} \equiv E[Y(0)]$ and variance $\sigma^2_{Y(0)} \equiv E[(Y(0) - \mu_{Y(0)})^2]$ are known.

Property of the driving process/noise $\{Z(n)\}$***:***

$\{Z(n)\}$ is a white noise with a possibly time-varying variance (non-stationary white noise) :

– $E[Z(n)] = 0$

– $E[Z(n)Z(n+k)] = \sigma^2_{ZZ}(n)\delta(k)$

Property of the feedback coefficients $\{h(n)\}$:

$\{h(n)\}$ is a known deterministic sequence.

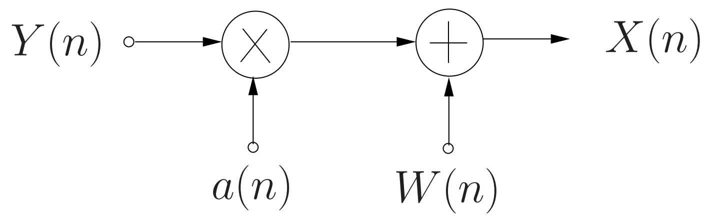
Remark: Provided $\{h(n)\}$ and $\{\sigma^2_{ZZ}(n)\}$ are constant and $\{Z(n)\}$ is a Gaussian random process, then $\{X(n)\}$ is an AR(1) process.

- **Observation (or channel) model**

The observable sequence $X(n)$ is given by

$$X(n) = a(n)Y(n) + W(n) \quad , n = 1, 2, ... \tag{5.2}$$

Block diagram representation of (5.2)



Property of the weighting sequence $\{a(n)\}$:

$\{a(n)\}$ is a known deterministic sequence.

Property of the noise $\{W(n)\}$:

$\{W(n)\}$ is a non-stationary white noise:

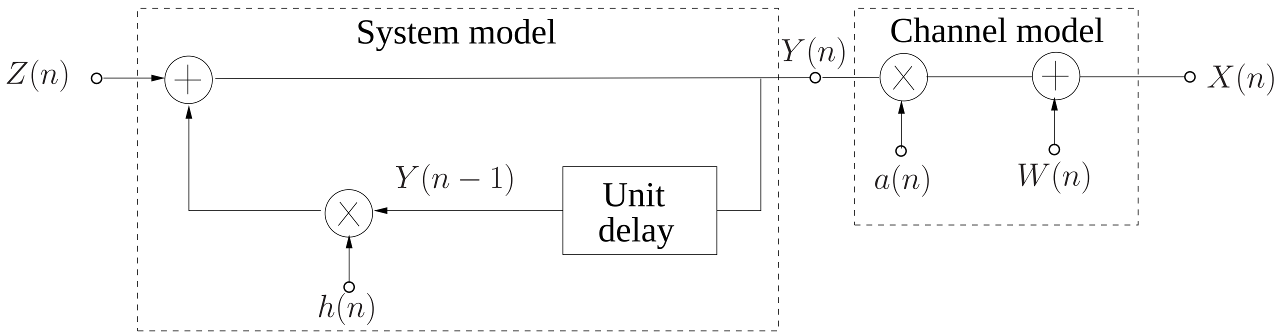
– $E[W(n)] = 0$

– $E[W(n)W(n+k)] = \sigma^2_{WW}(n)\delta(k)$

- **Additional "weak independence" assumptions**

  $Y(0)$, $\{Z(n)\}$, and $\{W(n)\}$ are uncorrelated.

- **Block diagram of the complete signal model**



5.1.2 Recursive implementation of the LMMSEE

- **Objective:**

  To find a ***recursive implementation***[1] of the LMMSEE of $Y(n)$ based on the observation of $X(1), ..., X(n)$.

- **Recursive implementation:**

  We need the following definitions:

  – $\hat{Y}(n \mid n) \equiv$ LMMSEE of $Y(n)$ based on the observation of $X(1), ..., X(n)$

  Estimation of $Y(n)$.

  – $\hat{Y}(n+1 \mid n) \equiv$ LMMSEE of $Y(n+1)$ based on the observation of $X(1), ..., X(n)$

  One-step prediction of $Y(n+1)$ ***at time*** n.

  – $\hat{X}(n+1 \mid n) \equiv$ LMMSEE of $X(n+1)$ based on the observation of $X(1), ..., X(n)$

  One-step prediction of $X(n+1)$.

Recursive implementation of the LMMSEE of $Y(n)$:

$$\underbrace{\hat{Y}(n+1 \mid n+1)}_{\substack{\text{Estimation}\\ \text{at time } n+1}} \equiv \mathcal{LF}(\underbrace{\hat{Y}(n \mid n)}_{\substack{\text{Estimation}\\ \text{at time } n}}, \underbrace{X(n+1)}_{\substack{\text{Observation}\\ \text{at time } n+1}})$$

where $\mathcal{LF}$ denotes a linear function to be found.

[1] See Section 5.3 for an example of a recursive estimator.

- **We shall know:**

  1. Such a recursive implementation of the LMMSEE exists. It is called the ***Kalman Filter***.

  2. The recursion is split into two steps:

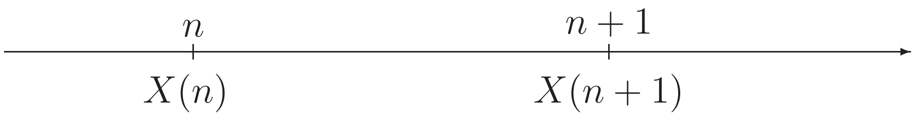
     – Step 1: ***One-step prediction***:

     $$P : \hat{Y}(n \mid n) \underset{P}{\longrightarrow} \hat{Y}(n+1 \mid n)$$

     – Step 2: ***Updating***:

     $$U : \hat{Y}(n+1 \mid n) \xrightarrow[U]{X(n+1)} \hat{Y}(n+1 \mid n+1)$$

Temporal evolution of the recursive estimation procedure in the Kalman filter:



$$\hat{Y}(n \mid n-1) \xrightarrow[U]{X(n)} \hat{Y}(n \mid n) \underset{P}{\longrightarrow} \hat{Y}(n+1 \mid n) \xrightarrow[U]{X(n+1)} \hat{Y}(n+1 \mid n+1) \underset{P}{\longrightarrow} \ldots$$

3. The mean-squared estimation error $E[(Y(n) - \hat{Y}(n \mid n))^2]$ can also be computed recursively.

   We shall need the following definitions:

   - $R(n\,|\,n) \equiv E[(Y(n) - \hat{Y}(n\,|\,n))^2]$

     $\equiv$ mean-squared estimation error at time n

   - $R(n+1\,|\,n) \equiv E[(Y(n+1) - \hat{Y}(n+1\,|\,n))^2]$

     $\equiv$ mean-squared one-step prediction error at time n

5.1.3 Derivation of the equations of the Kalman filter

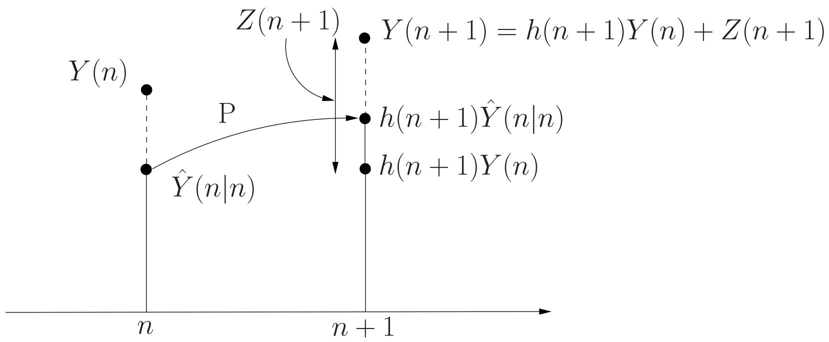
- **Prediction step:**

$$\hat{Y}(n+1 \mid n) = h(n+1)\hat{Y}(n \mid n) \quad (5.3)$$

$$R(n+1 \mid n) = h^2(n+1)R(n \mid n) + \sigma_{ZZ}^2(n+1) \quad (5.4)$$

Proof of (5.3):

(5.3) follows from the linearity property of the expectation.



Let us show that (5.3) satisfies the orthogonality principle (OP) and therefore is the LMMSEE:

Let $m = 1, ..., n$:

$$E[(Y(n+1) - \hat{Y}(n+1 \mid n))X(m)]$$

$$= E[(\overbrace{h(n+1)Y(n) + Z(n+1)} - \overbrace{h(n+1)\hat{Y}(n \mid n)})X(m)]$$

$$= h(n+1) \underbrace{E[(Y(n) - \hat{Y}(n \mid n))X(m)]}_{\substack{=0 \\ \text{OP for } \hat{Y}(n|n)}} + \underbrace{E[Z(n+1)X(m)]}_{\substack{=0 \\ Z(n+1) \text{ and } X(n) \\ \text{are uncorrelated}}}$$

$$= 0 \quad \surd$$

Proof of (5.4):

$$
\begin{aligned}
R(n+1 \mid n) &= E[(\qquad Y(n+1) \qquad - \qquad \hat{Y}(n+1 \mid n))^2] \\
&= E[(\overbrace{h(n+1)Y(n) + Z(n+1)} - \overbrace{h(n+1)\hat{Y}(n \mid n)})^2] \\
&= E[(\underbrace{h(n+1)[Y(n) - \hat{Y}(n \mid n)]} + \underbrace{Z(n+1)})^2]
\end{aligned}
$$

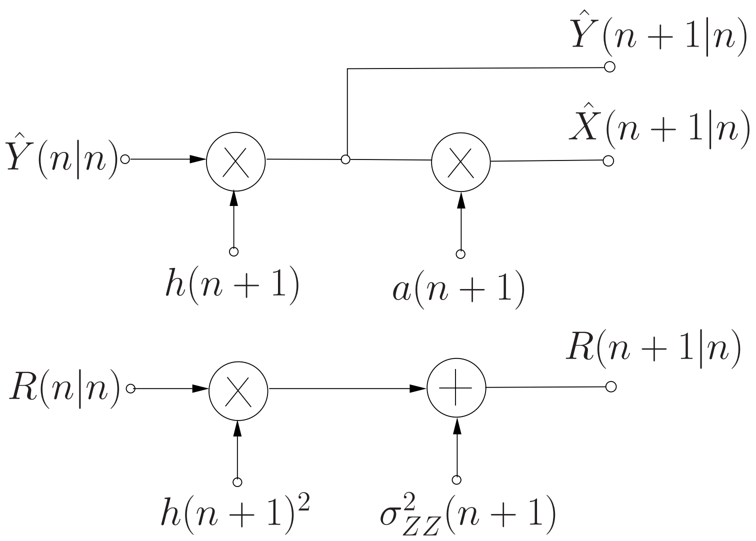
These two random variables are uncorrelated

$$
\begin{aligned}
&= h(n+1)^2 E[(Y(n) - \hat{Y}(n \mid n))^2] + E[Z(n+1)^2] \\
&= h(n+1)R(n \mid n) + \delta^2_{ZZ}(n+1)
\end{aligned}
$$

With the same argument as that used for the proof of (5.3) we show that

$$\boxed{\hat{X}(n+1 \mid n) = a(n+1)\hat{Y}(n+1 \mid n)}$$

Block diagram of the prediction step:



- **Updating step:**

$$\hat{Y}(n+1 \mid n+1) = \hat{Y}(n+1 \mid n) + b(n+1)[X(n+1) - \hat{X}(n+1 \mid n)] \quad (5.5)$$

$$R(n+1 \mid n+1) = [1 - b(n+1)a(n+1)]R(n+1 \mid n) \quad (5.6)$$

with

$$b(n+1) \equiv \frac{a(n+1)R(n+1 \mid n)}{a(n+1)^2 R(n+1 \mid n) + \sigma^2_{WW}(n+1)}$$

Interpretation of (5.5):

$$\hat{Y}(n+1 \mid n+1) = \underbrace{\hat{Y}(n+1 \mid n)}_{\substack{\text{One-step}\\ \text{prediction}\\ \text{of } Y(n+1)}} \underbrace{+\, b(n+1)\underbrace{\left[\overbrace{X(n+1)}^{\substack{\text{New}\\ \text{observation}}} - \overbrace{\hat{X}(n+1 \mid n)}^{\substack{\text{One-step}\\ \text{prediction}\\ \text{of } X(n+1)}}\right]}_{\substack{\text{Residual error}\\ \text{of } \hat{X}(n+1 \mid n)}}}_{\text{Correction factor}}$$

Kalman gain:

The coefficient $b(n)$ is called the ***Kalman gain*** of the filter.

Proof of (5.5) :

We seek an updating equation given by (5.5) and determine $b(n+1)$ so that (5.5) satisfies the orthogonality principle.

1st case: $m = 1, ..., n$

$$E[(Y(n+1) - \hat{Y}(n+1 \mid n+1))X(m)]$$

$$= \underbrace{E[(Y(n+1) - \hat{Y}(n+1 \mid n))X(m)]}_{\substack{=0 \\ \text{OP for } \hat{Y}(n+1 \mid n)}} - b(n+1) \underbrace{E[(X(n+1) - \hat{X}(n+1 \mid n))X(m)]}_{\substack{=0 \\ \text{OP for } \hat{X}(n+1 \mid n)}}$$

$$= 0 \quad \surd$$

2nd case: $m = n + 1$

$$\begin{aligned} &E[(Y(n+1) - \hat{Y}(n+1 \mid n+1))X(n+1)] \\ &\qquad = E[(Y(n+1) - \hat{Y}(n+1 \mid n))X(n+1)] \\ &\qquad\qquad - b(n+1)E[(X(n+1) - \hat{X}(n+1 \mid n))X(n+1)] \end{aligned}$$

We determine $b(n+1)$ such that the above expression vanishes:

$$b(n+1) = \frac{E[(Y(n+1) - \hat{Y}(n+1 \mid n))X(n+1)]}{E[(X(n+1) - \hat{X}(n+1 \mid n))X(n+1)]} = \frac{I}{II}$$

→ ***Computation of I:***

$$
\begin{aligned}
I &= E[(Y(n+1)-\hat{Y}(n+1\mid n))\overbrace{X(n+1)}]\\
&= E[(Y(n+1)-\hat{Y}(n+1\mid n))(a(n+1)Y(n+1)+W(n+1))]\\
&= a(n+1)\underbrace{E[(Y(n+1)-\hat{Y}(n+1\mid n))Y(n+1)]}_{\substack{=E[(Y(n+1)-\hat{Y}(n+1\mid n))^2]\\ \uparrow OP \Rightarrow E[(Y-\hat{Y})Y]=E[(Y-\hat{Y})^2]}}\\
&\quad + \underbrace{E[(Y(n+1)-\hat{Y}(n+1\mid n))W(n+1)]}_{=0}
\end{aligned}
$$

$Y(n+1)-\hat{Y}(n+1\mid n)$ and $W(n+1)$ are uncorrelated because $Y(n+1)-\hat{Y}(n+1\mid n)$ depends on $Y(0),Y(1),...,Y(n),Y(n+1),X(1),...,X(n)$

$$
\begin{aligned}
&= a(n+1)E[(Y(n+1)-\hat{Y}(n+1\mid n))^2]\\
I &= a(n+1)R(n+1\mid n)
\end{aligned}
$$

→ ***Computation of II:***

$$
\begin{aligned}
II &= E[(X(n+1)-\hat{X}(n+1\mid n))X(n+1)]\\
&= E[(\overbrace{X(n+1)}-\overbrace{\hat{X}(n+1\mid n)})^2] \quad \leftarrow OP\\
&= E[(a(n+1)Y(n+1)+W(n+1)-a(n+1)\hat{Y}(n+1\mid n))^2]\\
&= E[(a(n+1)\underbrace{(Y(n+1)-\hat{Y}(n+1\mid n))}+\underbrace{W(n+1))^2}]
\end{aligned}
$$

these random variables are uncorrelated

$$
\begin{aligned}
&= a(n+1)^2E[(Y(n+1)-\hat{Y}(n+1\mid n))^2]+E[W(n+1)^2]\\
II &= a(n+1)^2R(n+1\mid n)+\sigma^2_{WW}(n+1)
\end{aligned}
$$

Proof of (5.6):

$$R(n+1 \mid n+1) = E[(Y(n+1) - \hat{Y}(n+1 \mid n+1))^2]$$

$$\overset{OP}{=} E[(Y(n+1) - \hat{Y}(n+1 \mid n+1))Y(n+1)]$$

$$= \underbrace{E[(Y(n+1) - \hat{Y}(n+1 \mid n))Y(n+1)]}$$

$$\overset{OP}{=} E[(Y(n+1) - \hat{Y}(n+1 \mid n))^2] = R(n+1 \mid n)$$

$$- b(n+1)E[\underbrace{(X(n+1) - \hat{X}(n+1 \mid n))}Y(n+1)]$$

$$= a(n+1)Y(n+1) + W(n+1) - a(n+1)\hat{Y}(n+1 \mid n)$$

$$= a(n+1)(Y(n+1) - \hat{Y}(n+1 \mid n)) + W(n+1)$$

$$= R(n+1 \mid n)$$

$$- b(n+1)a(n+1)\,\underbrace{E[(Y(n+1) - \hat{Y}(n+1 \mid n))Y(n+1)]}$$

$$\overset{OP}{=} E[(Y(n+1) - \hat{Y}(n+1 \mid n))^2]$$

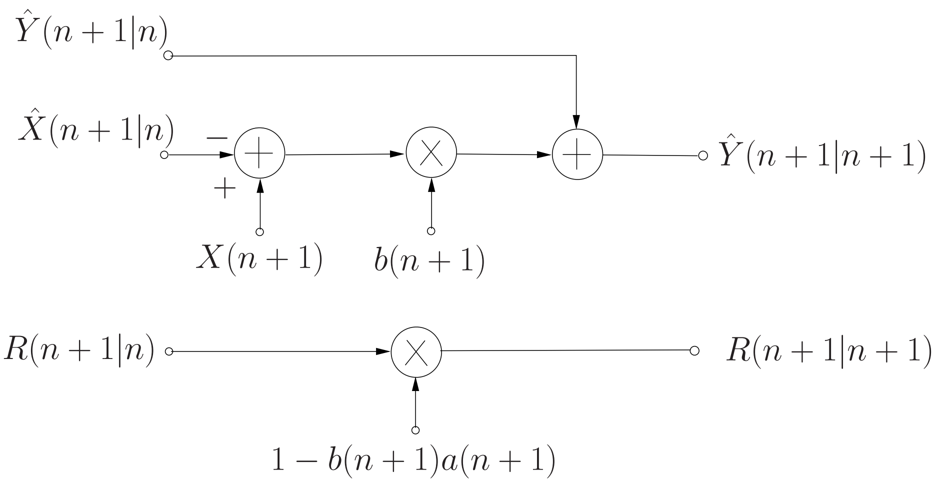
$$- b(n+1)E[\underbrace{Y(n+1)}\,\underbrace{W(n+1)}]$$

these random variables
are uncorrelated:
$E[Y(n+1)W(n+1)] = 0$

$$= R(n+1 \mid n) - b(n+1)a(n+1)R(n+1 \mid n)$$

$$R(n+1 \mid n) = [1 - b(n+1)a(n+1)]R(n+1 \mid n)$$

Block diagram of the updating step:

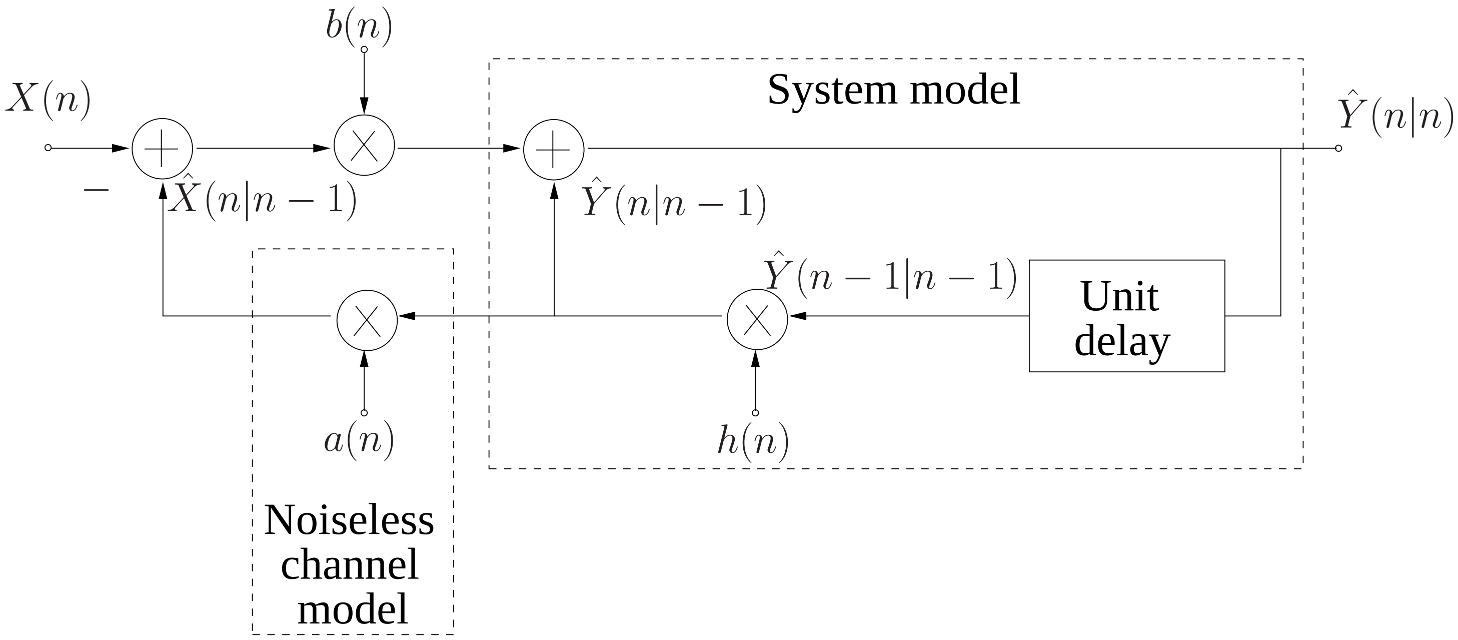


- **Initialization:**

$$\hat{Y}(0 \mid 0) = \mu_{Y(0)}$$

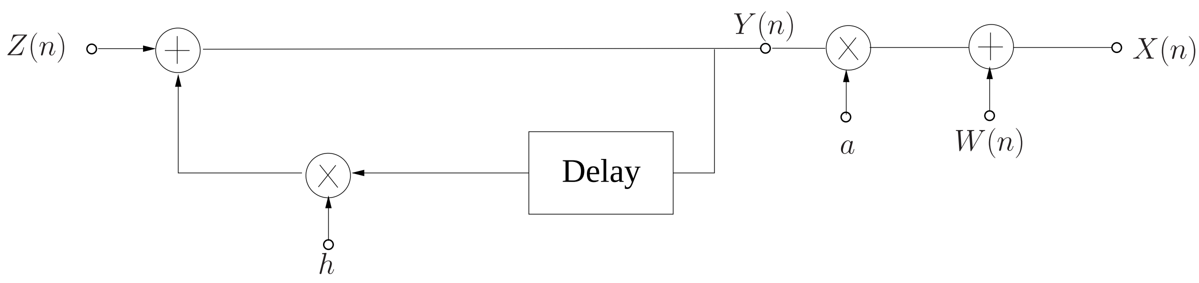
$$R(0 \mid 0) = \sigma^2_{Y(0)}$$

Block diagram of the Kalman filter:



5.1.4 Steady-state Kalman filter when the system and channel models are time-invariant

We consider the time-invariant system and channel models as depicted below:



The system-driving process $Z(n)$ and the channel noise $W(n)$ are uncorrelated (i.e white) wide-sense stationary process:

- $E[Z(n)Z(n+k)] = \sigma^2_{ZZ}(n)\delta(k)$
- $E[W(n)W(n+k)] = \sigma^2_{WW}\delta(k)$

Equations of the Kalman filter estimating $Y(n)$:

$$R(n+1 \mid n) = h^2 R(n \mid n) + \sigma^2_{ZZ}$$

$$b(n+1) = \frac{aR(n+1 \mid n)}{a^2 R(n+1 \mid n) + \sigma^2_{WW}}$$

$$R(n+1 \mid n+1) = [1 - ab(n+1)]R(n+1 \mid n)$$

For $n \to \infty$ the three sequences $\{R(n+1 \mid n)\}$, $\{b(n)\}$, and $\{R(n+1 \mid n+1)\}$ converge, i.e.

$$\left.\begin{aligned} R(n+1 \mid n) &\to R_p(\infty) \\ b(n) &\to b(\infty) \\ R(n+1 \mid n+1) &\to R(\infty) \end{aligned}\right\} \qquad n \to \infty$$

The Kalman filter converges to its steady state.

The above limits can be calculated by inserting them into the equations of the Kalman filter:

$$R_p(\infty) = h^2 R(\infty) + \sigma_{ZZ}^2 \tag{5.7}$$

$$b(\infty) = \frac{aR_p(\infty)}{a^2 R_p(\infty) + \sigma_{WW}^2} \tag{5.8}$$

$$R(\infty) = [1 - ab(\infty)]R_p(\infty) \tag{5.9}$$

Inserting (5.8) into (5.9), we obtain

$$R(\infty) = \left[1 - \frac{a^2 R_p(\infty)}{a^2 R_p(\infty) + \sigma_{WW}^2}\right] R_p(\infty)$$

$$= \frac{\sigma_{WW}^2 R_p(\infty)}{a^2 R_p(\infty) + \sigma_{WW}^2}$$

Substituting (5.7) into the last expression yields the so-called steady-state Ricatti equation

$$R(\infty) = \frac{\sigma_{WW}^2[h^2 R(\infty) + \sigma_{ZZ}^2]}{a^2[h^2 R(\infty) + \sigma_{ZZ}^2] + \sigma_{WW}^2}$$

The Ricatti equation is a quadratic equation that can be solved numerically. e.g. by using Newton's method.

Then, $R_p(\infty)$ and b(∞) follow by inserting the numerical solution for $R(\infty)$ into (5.7) and (5.8), respectively.

Example:

The steady-state solutions for the model with parameter setting

- $h = 0.9$
- $a = 0.1$
- $\sigma_{ZZ} = \sigma_{WW} = 1$

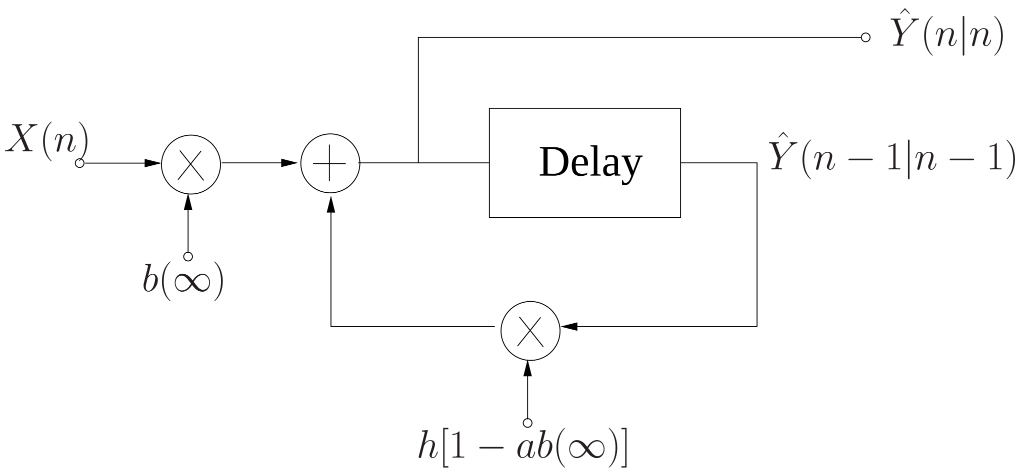
read

- $R_(\infty) = 0.5974$
- $R_p(\infty) = 1.4839$
- $b(\infty) = 0.5974$

Input-output relationship of the steady-state Kalman filter:

$$\hat{Y}(n \mid n) = h\hat{Y}(n-1 \mid n-1) + b(\infty)[X(n) - ah\hat{Y}(n-1 \mid n-1)]$$
$$= b(\infty)X(n) + h[1 - ab(\infty)]\hat{Y}(n-1 \mid n-1)$$

Block-diagram of the steady-state Kalman filter



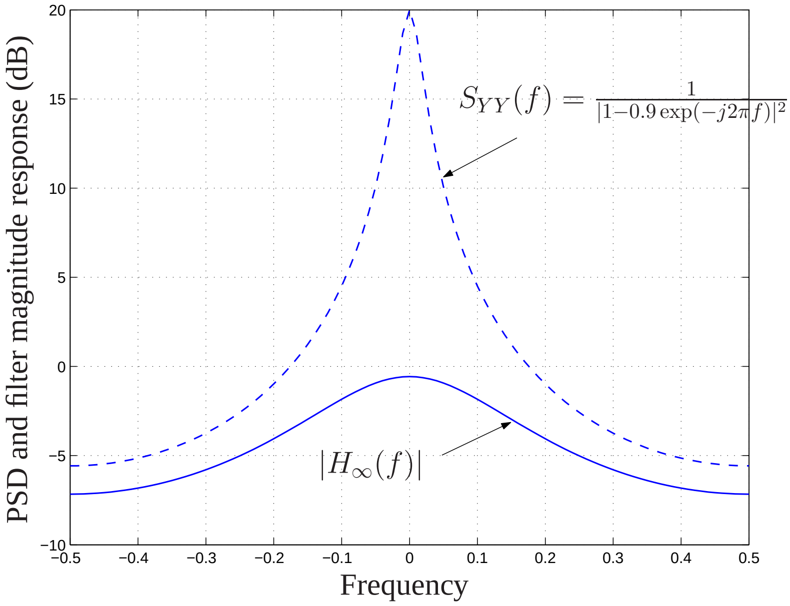
The steady-state Kalman filter is an infinite impulse response (IIR) filter with transfer function

$$H_\infty(f) = \frac{b_\infty}{1 - h[1-ab(\infty)]exp(-j2\pi f)}$$

$$H_\infty(z) = \frac{b_\infty}{1 - h[1-ab(\infty)]z^{-1}}$$

Example (cont'd):

$$H_\infty(f) = \frac{0.5974}{1 - 0.3623 \cdot \exp(-j2\pi f)}$$



Comment:

The steady-state Kalman filter calculates the LMMSEE of $Y(n)$ based on the observation of the sequence $\{X(n)\}$ in the time window $[n, n-1, n-2, \ldots]$

Hence, the steady-state Kalman filter implements the ***Causal Wiener filter***.

5.2 Vector Kalman Filter

5.2.1 Signal Model

- **System model**

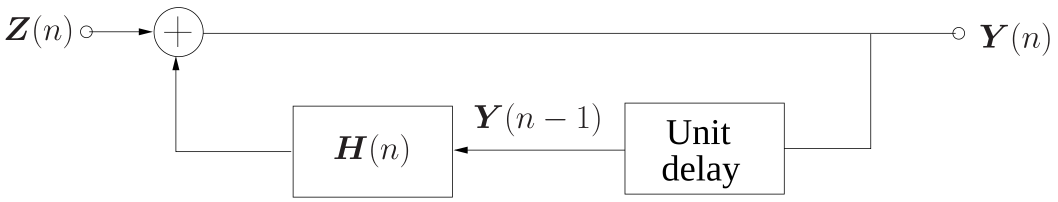
$$\mathbf{Y}(n) = \mathbf{H}(n)\mathbf{Y}(n-1) + \mathbf{Z}(n), \qquad n = 1, 2, \ldots \quad (5.10)$$

where:

  - $\mathbf{Y}(n) = [Y_1(n), ..., Y_r(n)]^{\mathrm{T}}$: r-dimensional (-D) random vector.
  - $\{\mathbf{Z}(n)\}$: r-D non-stationary white noise vector:
    - $E[\mathbf{Z}(n)] = \mathbf{0}$
    - $\sum_{\mathbf{Z}(n)\mathbf{Z}(n+k)} = \mathbf{Q}_{\mathbf{Z}}(n)\delta(k)$
  - $\{\mathbf{H}(n)\}$: sequence of known $r \times r$ matrices.

See the example discussed in Section 5.4.

Block diagram:



Initialization

$\mathbf{Y}(0)$ is a random vector specified by its expectation $\mu_{\mathbf{Y}(0)}$ and covariance matrix $\Sigma_{\mathbf{Y}(0)\mathbf{Y}(0)}$.

- **Observation Model**

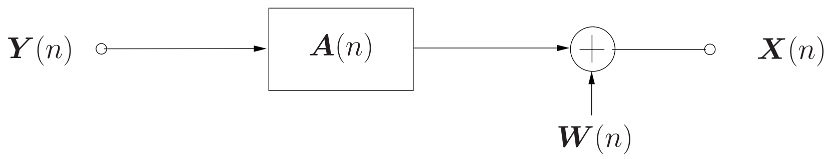
$$\mathbf{X}(n) = \mathbf{A}(n)\mathbf{Y}(n) + \mathbf{W}(n), n = 1, 2, ... \qquad (5.11)$$

where:

- $\mathbf{X}(n) = [X_1(n), ..., X_s(n)]^{\mathrm{T}}$: s-D random vector.
- $\{\mathbf{W}(n)\}$: s-D non-stationary white noise vector with auto-covariance

  $\Sigma_{\mathbf{W}(n)\mathbf{W}(n+k)} = \mathbf{Q}_{\mathbf{W}(n)}\delta(k)$
- $\{\mathbf{A}(n)\}$: sequence of known $s \times r$ matrices.

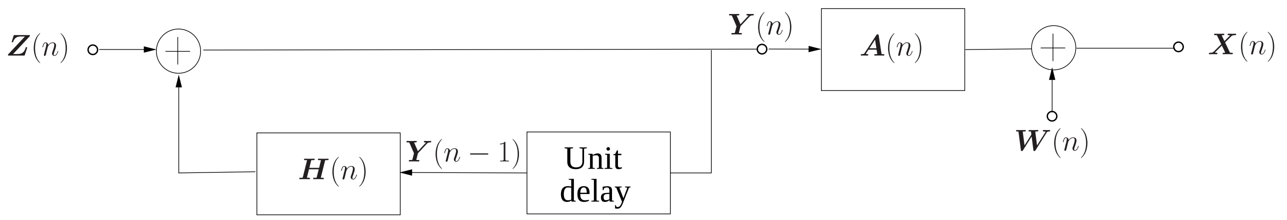
Block Diagram:



- **Additional independence assumption**

$\mathbf{Y}(0)$, $\{\mathbf{Z}(n)\}$, and $\{\mathbf{W}(n)\}$ are uncorrelated.

- **Complete Signal Model**



5.2.2 Equation of the vector Kalman filter

Let us define

- $\hat{\mathbf{Y}}(n \mid n) \quad \equiv \quad$ LMMSEE of $\quad \mathbf{Y}(n) \quad$ based on $\quad \mathbf{X}(1), ..., \mathbf{X}(n)$
- $\hat{\mathbf{Y}}(n+1 \mid n) \quad \equiv \quad$ LMMSEE of $\quad \mathbf{Y}(n+1) \quad$ based on $\quad \mathbf{X}(1), ..., \mathbf{X}(n)$
- $\hat{\mathbf{X}}(n+1 \mid n) \quad \equiv \quad$ LMMSEE of $\quad \mathbf{X}(n+1) \quad$ based on $\quad \mathbf{X}(1), ..., \mathbf{X}(n)$
- $\mathbf{R}(n \mid n) \quad \equiv \quad E[(\mathbf{Y}(n) - \hat{\mathbf{Y}}(n \mid n))(\mathbf{Y}(n) - \hat{\mathbf{Y}}(n|n))^{\mathrm{T}}]$
- $\mathbf{R}(n+1 \mid n) \quad \equiv \quad E[(\mathbf{Y}(n+1) - \hat{\mathbf{Y}}(n+1 \mid n))(\mathbf{Y}(n+1) - \hat{\mathbf{Y}}(n+1|n))^{\mathrm{T}}]$

We can apply the same reasoning as used for the scalar Kalman filter to show that the recursive equations of the vector Kalman filter are given as follows.

- **Recursive equations of the Kalman filter**

Prediction Step :

$$\begin{aligned}
\hat{\mathbf{Y}}(n+1 \mid n) &= \mathbf{H}(n+1)\hat{\mathbf{Y}}(n \mid n) \\
\hat{\mathbf{X}}(n+1 \mid n) &= \mathbf{A}(n+1)\hat{\mathbf{Y}}(n+1 \mid n) \\
\mathbf{R}(n+1 \mid n) &= \mathbf{H}(n+1)\mathbf{R}(n \mid n)\mathbf{H}(n+1)^{\mathrm{T}} + \mathbf{Q}_{\mathbf{Z}}(n+1)
\end{aligned}$$

Updating Step :

$$\hat{\mathbf{Y}}(n+1 \mid n+1) = \hat{\mathbf{Y}}(n+1 \mid n) + \mathbf{B}(n+1)[\mathbf{X}(n+1) - \hat{\mathbf{X}}(n+1 \mid n)]$$

$$\mathbf{R}(n+1 \mid n+1) = [\mathbf{I} - \mathbf{B}(n+1)\mathbf{A}(n+1)]\mathbf{R}(n+1 \mid n)$$

with the Kalman matrix

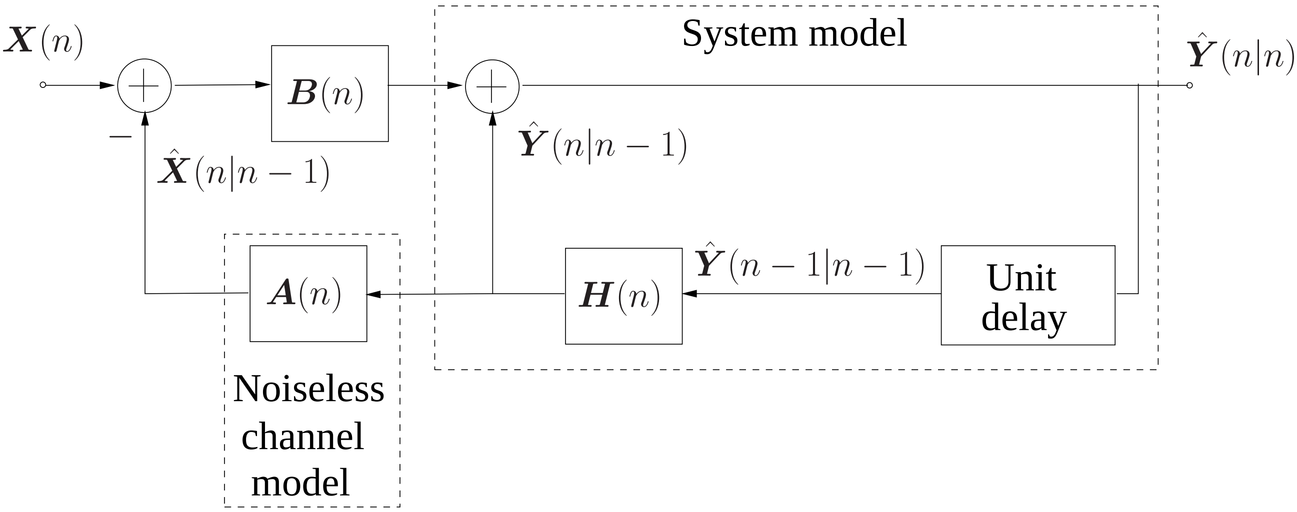
$$\mathbf{B}(n+1) \equiv \mathbf{R}(n+1 \mid n)\mathbf{A}(n+1)^{\mathrm{T}}[\mathbf{A}(n+1)\mathbf{R}(n+1 \mid n)\mathbf{A}(n+1)^{\mathrm{T}} + \mathbf{Q}_{\mathbf{W}}(n+1)]^{-1}$$

Initialization :

$$\hat{\mathbf{Y}}(0 \mid 0) = \mu_{\mathbf{Y}}(0)$$

$$\mathbf{R}(0 \mid 0) = \sum\nolimits_{\mathbf{Y}(0)\mathbf{Y}(0)}$$

- **Block diagram of the vector Kalman filter**



5.3 Example of a recursive estimator

- **Signal model**

$$X(n) = Y + W(n) \qquad n = 1, 2, 3, \ldots$$

Where:

  - Y is and unknown constant to be estimated based on the observation of $\{X(n)\}$.
  - $\{W(n)\}$ is a white noise sequence.

- **Arithmetic mean**

An appealing linear estimator for Y is the ***arithmetic mean***

$$\hat{Y}(n) = \frac{1}{n}\sum_{m=1}^{n} X(m)$$

Drawback: To compute $\hat{Y}(n)$ based on the above formula, $X(1), \ldots, X(n)$ need to be stored. The required memory grows linearly with n.

- **Recursive implementation**

$$\hat{Y}(n+1) = \frac{1}{n+1}\sum_{m=1}^{n} X(m) + \frac{1}{n+1}X(n+1)$$

$$\boxed{\hat{Y}(n+1) = \frac{n}{n+1}\hat{Y}(n) + \frac{1}{n+1}X(n+1)}$$

This estimator requires storage of one value, i.e. $\hat{Y}(n)$, only.

5.4 Example of a signal model: Target tracking

- **Equations of the movement of a target:**

Position: $U(t) = \int_0^t V(t')dt' + U(0)$, $U(0)$: initial position

Velocity: $V(t) = \int_0^t G(t')dt' + V(0)$, $V(0)$: initial velocity

Acceleration: $G(t)$ is assumed to be white noise.

- **Discrete-time model:**

$\frac{du}{dt}(t) = V(t)$ $\quad\quad$ $\frac{du}{dt}(nT_s) \approx U((n+1)T_s) - U(nT_s)$

$\frac{dv}{dt}(t) = G(t)$ $\quad\quad$ $\frac{dv}{dt}(nT_s) \approx G((n+1)T_s) - G(nT_s)$

$U((n+1)T_s) - U(nT_s) = V(nT_s) \cdot T_s$ $\quad\quad$ T_s: Sampling interval

$V((n+1)T_s) - V(nT_s) = \tilde{G}(nT_s) \cdot T_s$ $\quad\quad$ $\tilde{G}= \tilde{G}(t)$ low-pass filtered with bandwidth $\frac{1}{2T_s}$.

State model

$$\underbrace{\begin{bmatrix} U(nT_s) \\ V(nT_s) \end{bmatrix}}_{\boldsymbol{Y}(n)} = \underbrace{\begin{bmatrix} 1 & T_s \\ 0 & 1 \end{bmatrix}}_{\boldsymbol{H}(n)} \underbrace{\begin{bmatrix} U((n-1)T_s) \\ V((n-1)T_s) \end{bmatrix}}_{\boldsymbol{Y}(n-1)} + \underbrace{\begin{bmatrix} 0 \\ T_s\tilde{G}((n-1)T_s) \end{bmatrix}}_{\boldsymbol{Z}(n)}$$

with

$$\boldsymbol{Y}(0) = [U(0), V(0)]^{\mathrm{T}},$$

$$\boldsymbol{Q_Z}(n) = \begin{bmatrix} 0 & 0 \\ 0 & T_s^2 E[\tilde{G}(nT_s)^2] \end{bmatrix}.$$

Observation model

$$Y(n) = U(nT_s) + \underbrace{W(n)}_{\text{Measurement error}}$$

$$X(n) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\boldsymbol{A}(n)} Y(n) + W(n)$$

where $W(n)$ is white noise with variance σ_{WW}^2.