Stochastic Processes II (FP-7.5)
Problem set 10

Problem 10.1: Transformation for trend elimination

Solution

a) Transform $Y(n)$ into a WSS process

Let Δ denote the 1st order discrete derivative:

$$\begin{aligned}\Delta Y(n) &= Y(n) - Y(n-1) \\ &= an^2 + V(n) - a(n-1)^2 - V(n-1) \\ &= an^2 - a(n-1)^2 + V(n) - V(n-1) \\ &= a[n^2 - (n-1)^2] + V(n) - V(n-1) \\ &= a\underbrace{(2n-1)}_{\text{linear trend}} + V(n) - V(n-1).\end{aligned}$$

Applying the 2^{nd} order derivative yields

$$\begin{aligned}\Delta^2 Y(n) &= \Delta\big(\Delta Y(n)\big) \\ &= \Delta Y(n) - \Delta Y(n-1) \\ &= a(2n-1) + V(n) - V(n-1) - a[2(n-1)-1] - V(n-1) + V(n-2) \\ &= 2an - a - 2an + 3a + V(n) - 2V(n-1) + V(n-2) \\ &= 2a + V(n) - 2V(n-1) + V(n-2).\end{aligned}$$

Define

$$\begin{aligned}X(n) &\doteq \Delta^2 Y(n) \\ &= 2a + V(n) - 2V(n-1) + V(n-2).\end{aligned}$$

As a linear combination of the zero-mean WSS process $V(n-k), k = 0, 1, 2, 3$ and the constant value $2a$, $X(n)$ is a WSS process with mean $2a$. Therefore the 2^{nd} order discrete derivative (Δ^2) is the required transformation.

b) Autocorrelation function of $X(n)$:

Let

$$X(n) = 2a + Z(n),$$

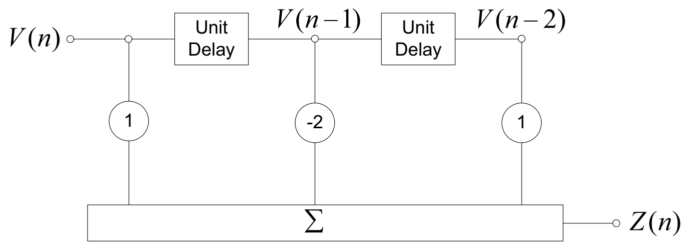
where

$$Z(n) \doteq V(n) - 2V(n-1) + V(n-2). \tag{1}$$

$Z(n)$ is a zero-mean WSS process. The autocorrelation function of $R_{XX}(k)$ can be written as

$$\begin{aligned}
R_{XX}(k) &= E\big[X(n)X(n+k)\big] \\
&= E\big[\big(2a+Z(n)\big)\big(2a+Z(n+k)\big)\big] \\
&= E\big[(2a)^2 + Z(n)\cdot 2a + Z(n+k)\cdot 2a + Z(n)Z(n+k)\big] \\
&= (2a)^2 + 2a\cdot \underbrace{E[Z(n)]}_{=0} + 2a\cdot \underbrace{E[Z(n+k)]}_{=0} + R_{ZZ}(k) \\
&= 4a^2 + R_{ZZ}(k). \qquad (2)
\end{aligned}$$

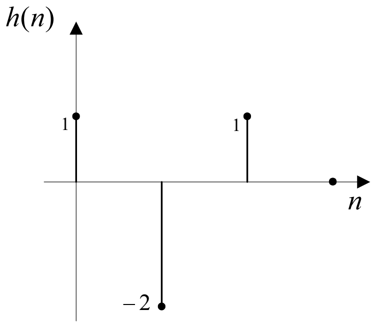
According to (1), $Z(n)$ is obtained by passing $V(n)$ through the following filter



Then

$$R_{ZZ}(k) = R_{VV}(k) * R_{hh}(k)$$

where $R_{hh}(k)$ is the autocorrelation function of the impulse response $h(n)$ of the filter.

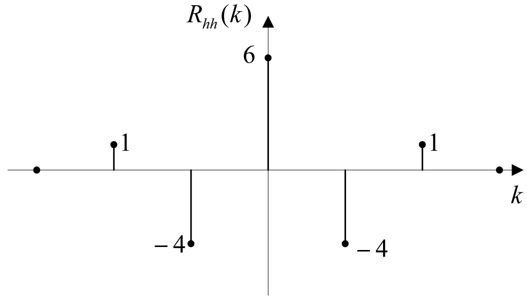


$$h(n) = \delta(n) - 2\delta(n-1) + \delta(n-2)$$

Hence,

$$\begin{aligned}
R_{hh}(k) &= \sum_{n=-\infty}^{\infty} h(n)h(n+k) \\
&= h(k) * h(-k) \\
&= \big[\delta(k) - 2\delta(k-1) + \delta(k-2)\big] * \big[\delta(-k) - 2\delta(-k-1) + \delta(-k-2)\big] \\
&= \delta(k+2) - 4\delta(k+1) + 6\delta(k) - 4\delta(k-1) + \delta(k-2).
\end{aligned}$$

Notice that in the above expression, $\delta(n-k_1) * \delta(n-k_2) = \delta(n-(k_1+k_2))$.



Then

$$
\begin{aligned}
R_{ZZ}(k) &= R_{VV}(k) * R_{hh}(k) \\
&= R_{VV}(k) * \big[\delta(k+2) - 4\delta(k+1) + 6\delta(k) - 4\delta(k-1) + \delta(k-2)\big] \\
&= R_{VV}(k+2) - 4R_{VV}(k+1) + 6R_{VV}(k) - 4R_{VV}(k-1) + R_{VV}(k-2).
\end{aligned}
$$

Inserting the above expression into (2) yields

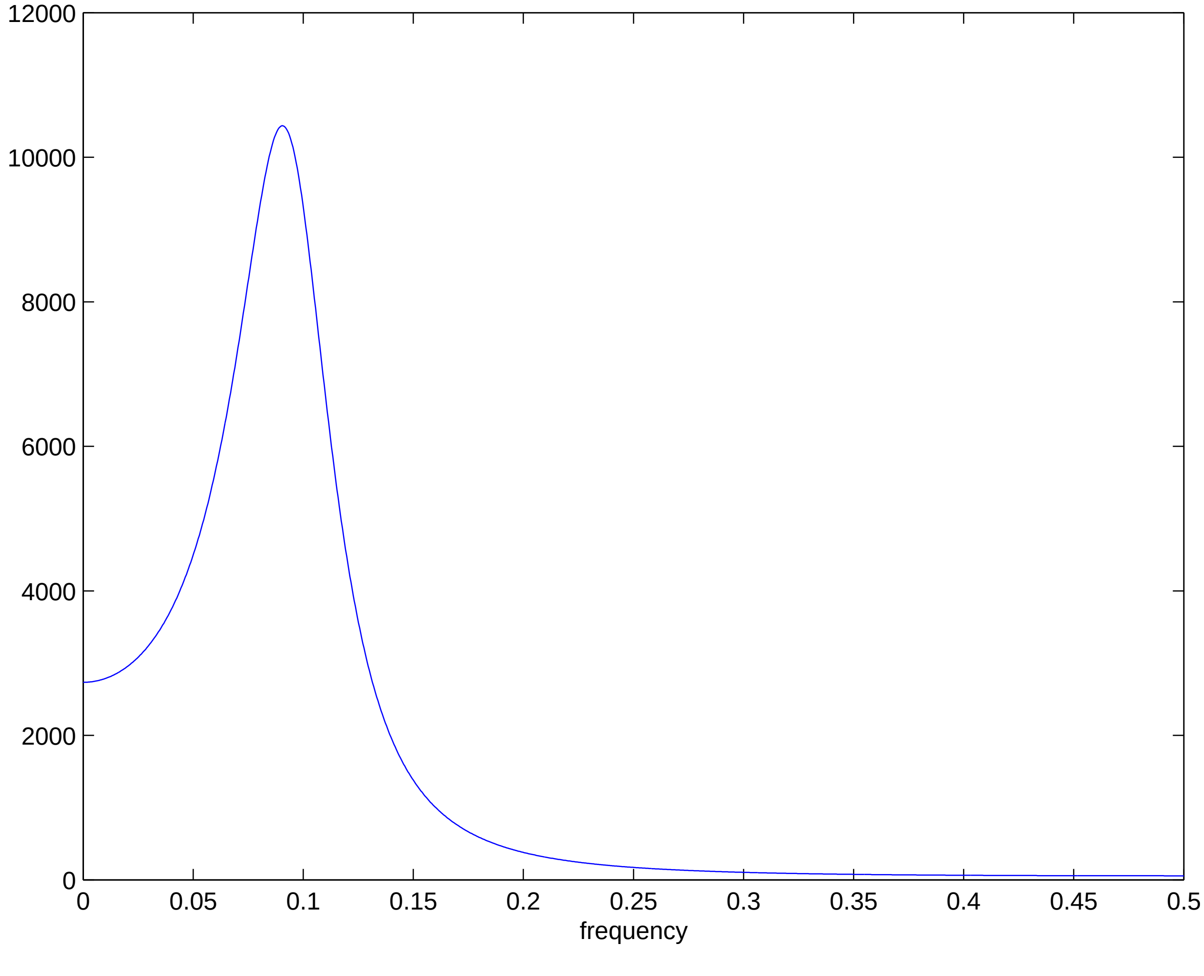
$$
\begin{aligned}
R_{XX}(k) &= 4a^2 + R_{ZZ}(k) \\
&= 4a^2 + R_{VV}(k+2) - 4R_{VV}(k+1) + 6R_{VV}(k) - 4R_{VV}(k-1) + R_{VV}(k-2).
\end{aligned}
$$

So $R_{XX}(k)$ is $R_{ZZ}(k)$ shifted by $4a^2$ upwards.

Problem 10.2: the sunspot data

The m file for finding the ARMA/AR coefficiencts can be downloaded from the course webpage.

From figure 1 and figure 2 we can see that the frequency resolution of ARMA(9,1) model is higher than the AR(3) model.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8000
6000
4000
2000
0
0
0.05
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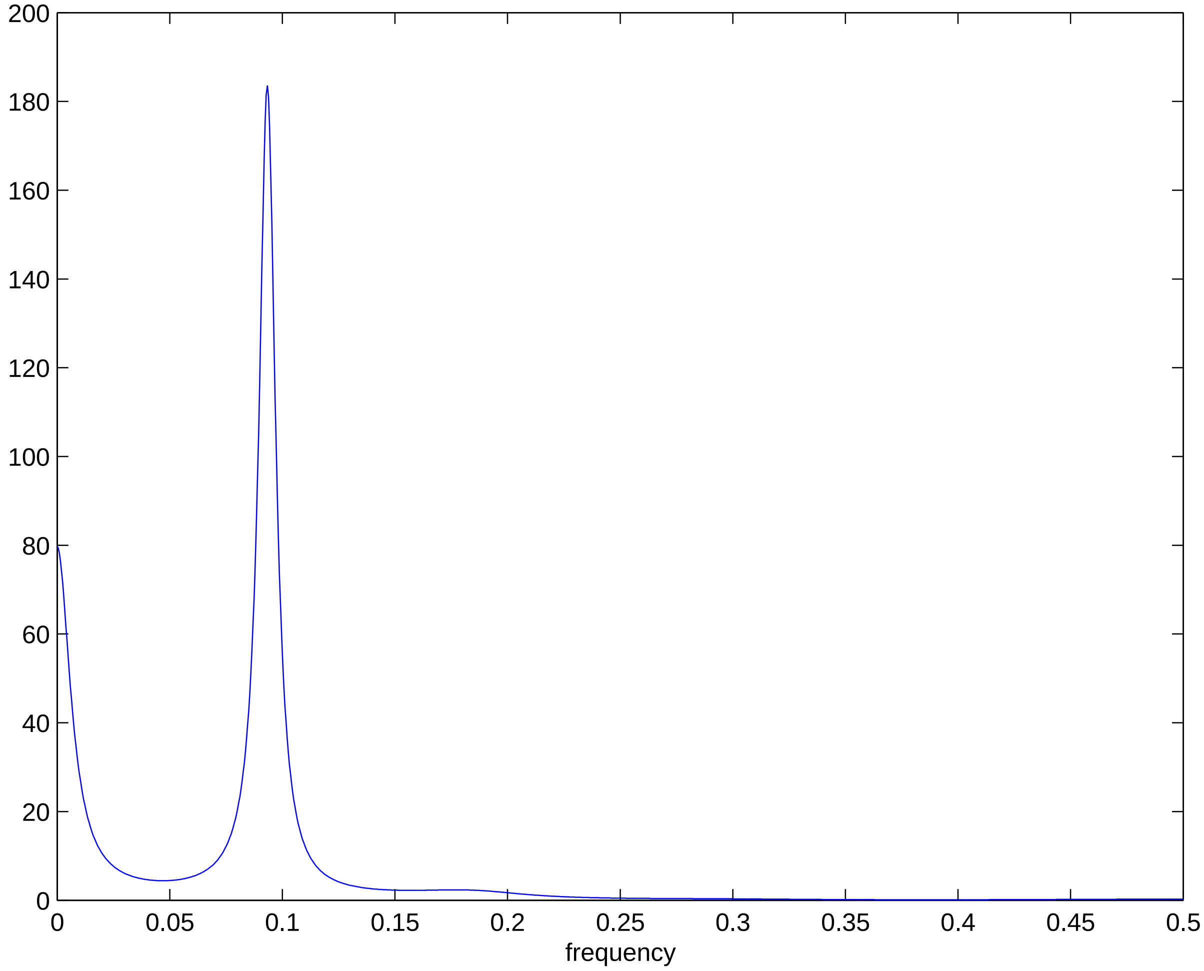
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
frequency

Figure 1: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160
140
120
100
80
60
40
20
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
frequency

Figure 2: