Stochastic Processes II (FP-7.5)
Solution Set 3

Problem 3.1 (Problem 6.2 in Shanmugan)

Solution:

The observed random variable Y is the number of "heads" showing up by tossing the coin eight times.

- Pdf of Y under both hypotheses:

$$
\begin{aligned}
H_0: \quad P[Y = n|H_0] &= \binom{8}{n} p_0^n (1-p_0)^{8-n} \quad \text{with} \quad p_0 = 0.5 \\
H_1: \quad P[Y = n|H_1] &= \binom{8}{n} p_1^n (1-p_1)^{8-n} \quad \text{with} \quad p_1 = 0.4
\end{aligned}
$$

- Likelihood ratio:

$$
\begin{aligned}
L(n) &= \frac{P[Y = n|H_1]}{P[Y = n|H_0]} \\
&= \frac{p_1^n (1-p_1)^{8-n}}{p_0^n (1-p_0)^{8-n}} \\
&= \left(\frac{p_1}{p_0}\right)^n \left(\frac{1-p_0}{1-p_1}\right)^n \left(\frac{1-p_1}{1-p_0}\right)^8 \\
L(n) &= \left(\frac{p_1}{p_0} \cdot \frac{1-p_0}{1-p_1}\right)^n \left(\frac{1-p_1}{1-p_0}\right)^8
\end{aligned}
$$

- Log-likelihood ratio:

$$
\begin{aligned}
l(n) &= \ln[L(n)] \\
&= n \cdot \left[\ln \frac{1-p_0}{p_0} - \ln \frac{1-p_1}{p_1}\right] + 8 \ln\left(\frac{1-p_1}{1-p_0}\right)
\end{aligned}
$$

- MAP decision rule:

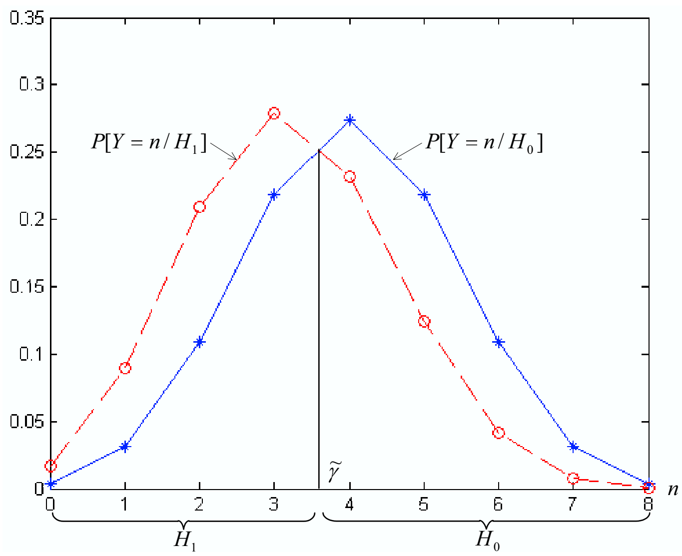
$$
\begin{aligned}
P[H_0] &= P[H_1] = \frac{1}{2} \\
\Rightarrow \gamma &= \frac{P[H_0]}{P[H_1]} = 1
\end{aligned}
$$

$$
\begin{aligned}
l(n) &\underset{H_0}{\overset{H_1}{\gtrless}} \ln\gamma \\
n\cdot\Bigg[\underbrace{\ln\frac{1-p_0}{p_0}}_{=0} - \ln\frac{1-p_1}{p_1}\Bigg] + 8\ln\left(\frac{1-p_1}{1-p_0}\right) &\underset{H_0}{\overset{H_1}{\gtrless}} 0 \\
n\cdot\ln\left(\frac{1-p_1}{p_1}\right) &\underset{H_0}{\overset{H_1}{\lessgtr}} 8\cdot\ln\left(\frac{1-p_1}{1-p_0}\right) \\
n &\underset{H_0}{\overset{H_1}{\lessgtr}} \underbrace{8\cdot\frac{\ln\left(\frac{1-p_1}{1-p_0}\right)}{\ln\left(\frac{1-p_1}{p_1}\right)}}_{=\tilde{\gamma}}
\end{aligned}
$$

Evaluation of the threshold $\tilde{\gamma}$:

$$
\begin{aligned}
\tilde{\gamma} &= 8\cdot\frac{\ln\left(\frac{1-p_1}{1-p_0}\right)}{\ln\left(\frac{1-p_1}{p_1}\right)} \\
&= 8\cdot\frac{\ln\left(\frac{0.6}{0.5}\right)}{\ln\left(\frac{0.6}{0.4}\right)} \\
&= 8\cdot\frac{\ln 1.2}{\ln 1.5} \\
&\approx 3.6
\end{aligned}
$$

$$
n \underset{H_0}{\overset{H_1}{\lessgtr}} 8\cdot\frac{\ln 1.2}{\ln 1.5}
$$



- Probability of incorrect decision P_e:

$$\begin{aligned} P_e &= P[D = H_1|H_0] \cdot P[H_0] + P[D = H_0|H_1] \cdot P[H_1] \\ &= 0.5 \cdot \left\{ P[D = H_1|H_0] + P[D = H_0|H_1] \right\} \end{aligned}$$

$$\begin{aligned} P[D = H_1|H_0] &= P[Y < 3.6|H_0] \\ &= \sum_{n=0}^{3} \binom{8}{n} p_0^n (1 - p_0)^{8-n} \\ &\approx 0.3633 \end{aligned}$$

$$\begin{aligned} P[D = H_0|H_1] &= P[Y > 3.6|H_1] \\ &= \sum_{n=4}^{8} \binom{8}{n} p_1^n (1 - p_1)^{8-n} \\ &\approx 0.4059 \end{aligned}$$

$$\begin{aligned} P_e &\approx 0.5 \cdot (0.3633 + 0.4059) \\ P_e &\approx 0.3846 \end{aligned}$$

Problem 3.2 (Problem 6.8 in Shanmugan)

Solution:

- Signal model:

$$Y = x + W$$

  where

  · $x = \begin{cases} 1 & ; \quad H_0 \\ -1 & ; \quad H_1 \end{cases}$

  · $W \sim \mathcal{N}(0, 1)$.

- Pdf of Y under H_0 and H_1:

$$\begin{aligned} f(y|H_0) &= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(y-1)^2\} \\ f(y|H_1) &= \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(y+1)^2\} \end{aligned}$$

- Log-likelihood ratio:

$$\begin{aligned} l(y) &= \ln \frac{f(y|H_1)}{f(y|H_0)} \\ &= -\frac{1}{2}(y+1)^2 + \frac{1}{2}(y-1)^2 \\ &= \frac{1}{2}\big[(y-1)^2 - (y+1)^2\big] \\ &= \frac{1}{2}\big[y^2 - 2y + 1 - y^2 - 2y - 1\big] \end{aligned}$$

$$l(y) = -2y$$

a. MAP decision rule:

$$\begin{aligned} l(y) &\underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P[H_0]}{P[H_1]}\right) \\ -2y &\underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{1/3}{2/3}\right) \\ -2y &\underset{H_0}{\overset{H_1}{\gtrless}} -\ln 2 \\ y &\underset{H_0}{\overset{H_1}{\lessgtr}} \underbrace{\frac{1}{2}\ln 2}_{=\tilde{\gamma}_{\text{MAP}}} \approx 0.35 \end{aligned}$$

b. Bayes decision rule:

$$
\begin{aligned}
l(y) &\underset{H_0}{\overset{H_1}{\gtrless}} \ln\Big(\frac{P[H_0](C_{10}-C_{00})}{P[H_1](C_{01}-C_{11})}\Big)\\
-2y &\underset{H_0}{\overset{H_1}{\gtrless}} \ln\Big(\frac{\frac{1}{3}(1-0)}{\frac{2}{3}(6-0)}\Big)\\
-2y &\underset{H_0}{\overset{H_1}{\gtrless}} \ln(\frac{1}{12})\\
-2y &\underset{H_0}{\overset{H_1}{\gtrless}} -\ln 12\\
y &\underset{H_0}{\overset{H_1}{\lessgtr}} \underbrace{\frac{1}{2}\ln 12}_{=\tilde{\gamma}_B} \approx 1.2425
\end{aligned}
$$

Value of $\bar{C}_{\min}$:

$$
\begin{aligned}
\bar{C}_{\min} &= 1\cdot\underbrace{P[D=H_1|H_0]}_{P_f}\cdot\frac{1}{3}+6\cdot\underbrace{P[D=H_0|H_1]}_{P_m}\cdot\frac{2}{3}\\
&= \frac{1}{3}[P_f+12P_m]
\end{aligned}
$$

$$
\begin{aligned}
P_f &= P[Y<\tilde{\gamma}_B|H_0]\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\tilde{\gamma}_B}\exp\big(-\frac{1}{2}(y-1)^2\big)dy\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\tilde{\gamma}_B-1}\exp(-\frac{1}{2}z^2)dz\\
&= 1-\frac{1}{\sqrt{2\pi}}\int_{\tilde{\gamma}_B-1}^{+\infty}\exp(-\frac{1}{2}z^2)dz\\
&= 1-Q(\tilde{\gamma}_B-1)\\
&= 1-Q(\frac{\ln 12}{2}-1)\\
&\approx 1-Q(0.24)\\
&\approx 0.5958.
\end{aligned}
$$

In the above expressions, $Q(u)$ is the Q-function $Q(u) = \frac{1}{\sqrt{2\pi}} \int\limits_{u}^{+\infty} \exp(-\frac{u^2}{2})du$.

$$\begin{aligned}
P_m &= P[Y > \tilde{\gamma}_B | H_1] \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{\tilde{\gamma}_B}^{\infty} \exp(-\frac{1}{2}(y+1)^2)dy \\
&= \frac{1}{\sqrt{2\pi}} \int\limits_{\tilde{\gamma}_B+1}^{\infty} \exp(-\frac{1}{2}z^2)dz \\
&= Q(\tilde{\gamma}_B + 1) \\
&= Q(\frac{\ln 12}{2} + 1) \\
&\approx Q(2.24) \\
&\approx 0.0125
\end{aligned}$$

$$\begin{aligned}
\bar{C}_{\min} &= \frac{1}{3}(P_f + 12 \cdot P_m) \\
&= \frac{1}{3}[0.5958 + 12 \cdot 0.0125] \\
\bar{C}_{\min} &\approx 0.2485
\end{aligned}$$