Stochastic Processes II (FP-7.5) Solution Set 4

Problem 4.1 (Problem 6.13 in Shanmugan)

Solution:

a. The MAP decision rule with multiple observations:

$$\frac{P(H_1|\mathbf{y})}{P(H_0|\mathbf{y})} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

By using Bayes' rule:

$$P(H_i|\mathbf{y}) = \frac{f_{\mathbf{Y}|H_i}(\mathbf{y}|H_i)P(H_i)}{f_{\mathbf{Y}}(\mathbf{y})}$$

The observations are independent and therefore

$$f_{\mathbf{Y}|H_1}(\mathbf{y}|H_1) = \prod_{i=1}^{2} \frac{1}{4} \exp\left(\frac{-y_i}{4}\right) = \frac{1}{16} \exp\left(\frac{-(y_1+y_2)}{4}\right)$$
$$f_{\mathbf{Y}|H_0}(\mathbf{y}|H_0) = \prod_{i=1}^{2} \frac{1}{2} \exp\left(\frac{-y_i}{2}\right) = \frac{1}{4} \exp\left(\frac{-(y_1+y_2)}{2}\right)$$

The priors are equally likely, i.e. $P(H_0) = P(H_1)$. Therefore the MAP decision rule is equivalent to the ML decision rule:

$$\frac{f_{\mathbf{Y}|H_1}(\mathbf{y}|H_1)P(H_1)}{f_{\mathbf{Y}|H_0}(\mathbf{y}|H_0)P(H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \Leftrightarrow \frac{f_{\mathbf{Y}|H_1}(\mathbf{y}|H_1)}{f_{\mathbf{Y}|H_0}(\mathbf{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \Leftrightarrow$$
$$\frac{\frac{1}{16}\exp\left(\frac{-(y_1+y_2)}{4}\right)}{\frac{1}{4}\exp\left(\frac{-(y_1+y_2)}{2}\right)} \underset{H_0}{\overset{H_1}{\gtrless}} 1 \Leftrightarrow \exp\left(\frac{y_1+y_2}{4}\right) \underset{H_0}{\overset{H_1}{\gtrless}} 4 \Leftrightarrow$$
$$y_1 + y_2 \underset{H_0}{\overset{H_1}{\gtrless}} 4\ln(4) \Leftrightarrow y_1 + y_2 \underset{H_0}{\overset{H_1}{\gtrless}} 5.5452$$

b. We use the decision variable $Z = Y_1 + Y_2$. Since the observation variable Z is the sum of two independent random variables Y_1 and Y_2 its density function is given by the convolution of the density functions of Y_1 and Y_2.

$$f_{Z|H_i}(z|H_i) = f_{Y_1|H_i}(z|H_i) * f_{Y_2|H_i}(z|H_i) \quad (i = 0, 1)$$

$$\begin{aligned}
f_{Z|H_1}(z|H_1) &= f_{Y_1|H_1}(z|H_1) * f_{Y_2|H_1}(z|H_1) \\
&= \int_0^z \frac{1}{4} \exp\left(\frac{-y}{4}\right) \frac{1}{4} \exp\left(\frac{-(z-y)}{4}\right) dy \\
&= \frac{1}{16} \int_0^z \exp\left(\frac{-z}{4}\right) dy \\
&= \frac{1}{16} \int_0^z \exp\left(\frac{-z}{4}\right) dy \\
&= \frac{1}{16} \exp\left(\frac{-z}{4}\right) \int_0^z dy \\
&= \frac{1}{16} z \exp\left(\frac{-z}{4}\right)
\end{aligned}$$

$$\begin{aligned}
f_{Z|H_0}(z|H_0) &= f_{Y_1|H_0}(z|H_0) * f_{Y_2|H_0}(z|H_0) \\
&= \int_0^z \frac{1}{2} \exp\left(\frac{-y}{2}\right) \frac{1}{2} \exp\left(\frac{-(z-y)}{2}\right) dy \\
&= \frac{1}{4} \int_0^z \exp\left(\frac{-z}{2}\right) dy \\
&= \frac{1}{4} \int_0^z \exp\left(\frac{-z}{2}\right) dy \\
&= \frac{1}{4} \exp\left(\frac{-z}{2}\right) \int_0^z dy \\
&= \frac{1}{4} z \exp\left(\frac{-z}{2}\right)
\end{aligned}$$

The probability of a miss is given by:

$$
\begin{aligned}
P_M = P(D_0|H_1) &= \int\limits_0^{4\ln(4)} f_{Z|H_1}(z|H_1)dz \\
&= \frac{1}{16} \int\limits_0^{4\ln(4)} z \exp\left(\frac{-z}{4}\right) dz \\
&= \frac{1}{4} \int\limits_0^{4\ln(4)} \frac{z}{4} \exp\left(\frac{-z}{4}\right) dz
\end{aligned}
$$

Substitution with $s = z/4$ yields

$$
\begin{aligned}
P_M = P(D_0|H_1) &= \int\limits_0^{\ln(4)} s \exp\left(-s\right) ds \\
&= \left[-\exp\left(-s\right)\left(s+1\right)\right]_0^{\ln(4)} \\
&= -\frac{1}{4}\left(\ln(4)+1\right)+1 \\
&\approx 0.403
\end{aligned}
$$

The probability of a false is given by:

$$
\begin{aligned}
P_F = P(D_1|H_0) &= \int\limits_{4\ln(4)}^{\infty} f_{Z|H_0}(z|H_0)dz \\
&= \frac{1}{4} \int\limits_{4\ln(4)}^{\infty} z \exp\left(\frac{-z}{2}\right) dz \\
&= \frac{1}{2} \int\limits_{4\ln(4)}^{\infty} \frac{z}{2} \exp\left(\frac{-z}{2}\right) dz
\end{aligned}
$$

Substitution with $s = z/2$ yields

$$
\begin{aligned}
P_F = P(D_1|H_0) &= \int\limits_{2\ln(4)}^{\infty} s \exp(-s)\, ds \\
&= \left[-\exp(-s)(s+1)\right]_{2\ln(4)}^{\infty} \\
&= \exp(-2\ln(4))(2\ln(4)+1) \\
&\approx 0.236
\end{aligned}
$$

Problem 4.2 (Problem 6.14 in Shanmugan)

Solution:

- Signal model:

$$Y(t) = x(t) + W(t), \quad t \in [0, T]$$

  where

  · $x(t) = \begin{cases} s_1(t) \text{ under } H_1 \\ s_0(t) \text{ under } H_0 \end{cases}$

  · $W(t)$ is white Gaussian noise with spectral height

$$\frac{N_0}{2} = 10^{-3} \text{ W/Hz}$$

  · $P[H_0] = P[H_1] = \frac{1}{2}$

- General formula for the decision rule (See page 3-20 of the lecture notes):

$$\int_0^T y(t)\big[s_1(t) - s_0(t)\big]\,dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln(\gamma) + \frac{1}{2}(E_{s_1} - E_{s_0}) \qquad (1)$$

a. The decision rule minimizing P_e is the MAP decision rule, i.e. Equation (1) with $\gamma = P[H_0]/P[H_1]$.

   In the particular case considered here,

   · $\gamma = 1$

   · $s_0(t) = -s_1(t)$

$$\Rightarrow \int_0^T y(t)\big[s_1(t) - s_0(t)\big]\,dt \; = 2\int_0^T y(t)s_1(t)dt$$

· $E_{s_0} = E_{s_1}$

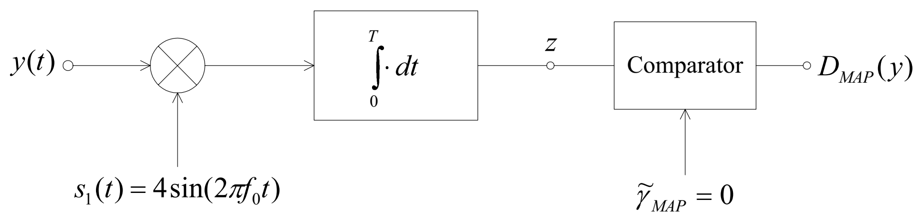
$$
\begin{aligned}
E_{s_0} = E_{s_1} &= \int_0^T \left[4\sin(2\pi f_0 t)\right]^2 dt \\
&= 16 \int_0^T \sin^2(2\pi f_0 t) dt \\
&= 16 \int_0^T \frac{1}{2}\left[1 - \cos\big(2\pi(2f_0)t\big)\right] dt \\
&= 8\Big[\underbrace{\int_0^T dt}_{T} - \underbrace{\int_0^T \cos\big(2\pi(2f_0)t\big) dt}_{=0}\Big] \\
E_{s_0} = E_{s_1} &= 8T
\end{aligned}
$$

MAP decision rule:

$$\int_0^T y(t)s_1(t)dt \underset{H_0}{\overset{H_1}{\gtrless}} 0, \qquad (2)$$

hence $\tilde{\gamma}_{MAP} = 0$.

Implementation of (2):



b. Computation of P_e:

Let us define

$$
\begin{aligned}
Z &\doteq \int_{-\infty}^{+\infty} Y(t)\left[s_1(t) - s_0(t)\right] dt \\
&= 2 \int_{-\infty}^{+\infty} Y(t)s_1(t)dt \\
&= 8 \int_{-\infty}^{+\infty} Y(t)\sin(2\pi f_0 t)dt.
\end{aligned}
$$

From Problem 6.4 we know that

$$\begin{aligned}
&\text{Under } H_0: \quad Z \sim \mathcal{N}(\mu_0, \sigma^2)\\
&\text{Under } H_1: \quad Z \sim \mathcal{N}(\mu_1, \sigma^2)
\end{aligned}$$

where

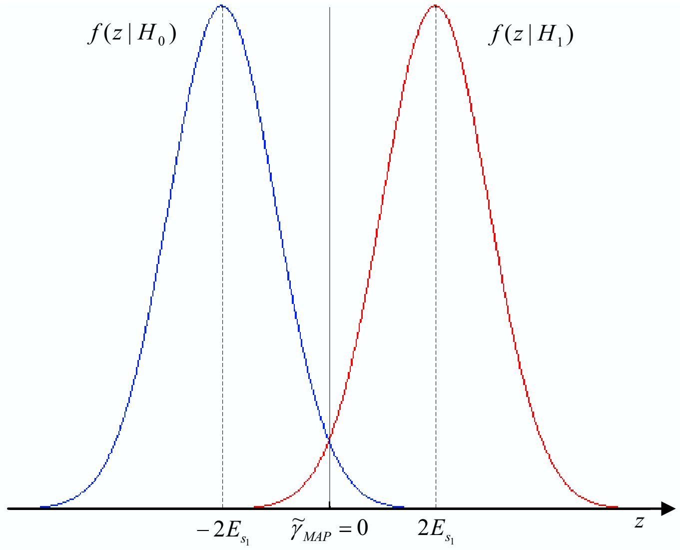
$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2} \int_{-\infty}^{+\infty} \big[s_1(t) - s_0(t)\big]^2 dt,\\
\mu_0 &= \int_{-\infty}^{+\infty} s_0(t)\big[s_1(t) - s_0(t)\big] dt,\\
\mu_1 &= \int_{-\infty}^{+\infty} s_1(t)\big[s_1(t) - s_0(t)\big] dt.
\end{aligned}$$

Computation of μ_0, μ_1, and σ^2:
We use the fact that $s_0(t) = -s_1(t)$,

$$\begin{aligned}
\sigma^2 &= \frac{N_0}{2} \int_{-\infty}^{+\infty} \big[2s_1(t)\big]^2 dt\\
&= 2N_0 E_{s_1}\\
&= 16 N_0 T\\
&= 16 \cdot 2 \cdot 10^{-3} \cdot 10^{-3}\\
&= 32 \cdot 10^{-6},
\end{aligned}$$

$$\begin{aligned}
\mu_0 &= -\int_{-\infty}^{+\infty} s_1(t)[2s_1(t)] dt\\
&= -2E_{s_1}\\
&= -16T\\
&= -16 \cdot 10^{-3},
\end{aligned}$$

$$\begin{aligned}
\mu_1 &= \int_{-\infty}^{+\infty} s_1(t)[2s_1(t)] dt\\
&= 2E_{s_1}\\
&= 16T\\
&= 16 \cdot 10^{-3}.
\end{aligned}$$



Computation of P_e

$$
\begin{aligned}
P_e &= P[z>0|H_0]\cdot\frac{1}{2}+P[z<0|H_1]\cdot\frac{1}{2}\\
&= \frac{1}{2}\cdot 2P[z>0|H_0] \qquad \big(P[z>0|H_0]=P[z<0|H_1]\big)\\
&= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\exp\{-\frac{1}{2\sigma^2}(z+2E_{s_1})^2\}dz\\
&= \frac{1}{\sqrt{2\pi}}\int_{\frac{2E_{s_1}}{\sigma}}^{+\infty}\exp\{-\frac{1}{2}u^2\}du \qquad \left(u=\frac{z+2E_{s_1}}{\sigma}\right)\\
&= Q(\frac{2E_{s_1}}{\sigma}).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\frac{2E_{s_1}}{\sigma} &= \frac{2\cdot E_{s_1}}{\sqrt{2N_0E_{s_1}}}\\
&= \sqrt{\frac{2E_{s_1}}{N_0}}\\
&= \sqrt{\frac{E_{s_1}}{N_0/2}}\\
&= \sqrt{\frac{8T}{N_0/2}}\\
&= \sqrt{\frac{8\cdot 10^{-3}}{10^{-3}}}\\
&= \sqrt{8}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
P_e &= Q(\sqrt{8}) \\
&\approx 0.0023
\end{aligned}
$$