Stochastic Processes II (FP-7.5) Solution Set 5

Problem 5.1 (Problem 7.5 in Shanmugan)

Solution:

- Signal model:

  $$X = mY + W$$

  where

  - m is a known constant;
  - $W \sim \mathcal{N}(\mu_W, \sigma_W^2)$;
  - Y is a random variable;
  - W and Y are independent.

- LMMSEE:

  $$\hat{Y} = a + bX.$$

  Hence

  $$\boldsymbol{h} = [a, b]^T.$$

  The coefficients are obtained from the equations (see page 4-5 of the lecture notes)

  $$\begin{aligned} \boldsymbol{h}^- &= \left(\Sigma_{XX}\right)^{-1}\Sigma_{XY} && (1) \\ h_0 &= \mu_Y - (\boldsymbol{h}^-)^T\mu_X. && (2) \end{aligned}$$

  Here:

  · $\boldsymbol{h}^- = b, h_0 = a$;

  · Σ_{XX}:

  $$\begin{aligned} \Sigma_{XX} &= \sigma_X^2 \\ &= m^2\sigma_Y^2 + \sigma_W^2; \end{aligned}$$

  · Σ_{XY}:

  $$\begin{aligned} \Sigma_{XY} &= E\big[(X-\mu_X)(Y-\mu_Y)\big] \\ &= E\big\{[mY + W - (m\mu_Y + \mu_W)](Y-\mu_Y)\big\} \\ &= E\big\{[m(Y-\mu_Y) + W - \mu_W](Y-\mu_Y)\big\} \\ &= mE[(Y-\mu_Y)^2] + \underbrace{E[(W-\mu_W)(Y-\mu_Y)]}_{=E[W-\mu_W]\cdot E[Y-\mu_Y]=0} \\ &= m\sigma_Y^2; \end{aligned}$$

  · $\mu_X = m\mu_Y + \mu_W$.

Insertion in (1) and (2) yields

$$\begin{aligned} b &= \frac{m\sigma_Y^2}{m^2\sigma_Y^2 + \sigma_W^2} \\ &= \frac{1}{m} \cdot \gamma, \end{aligned}$$

where

$$\gamma = \frac{1}{1 + \frac{\sigma_W^2}{m^2\sigma_Y^2}},$$

and

$$\begin{aligned} a &= \mu_Y - \frac{\gamma}{m}(m\mu_Y + \mu_W) \\ &= (1-\gamma)\mu_Y - \frac{\gamma}{m}\mu_W. \end{aligned}$$

So that the LMMSEE is given by

$$\hat{Y} = \left[(1-\gamma)\mu_Y - \frac{\gamma}{m}\mu_W\right] + \frac{\gamma}{m}X$$

with

$$\gamma = \left(1 + \frac{\sigma_W^2}{m^2\sigma_Y^2}\right)^{-1}.$$

Problem 5.2 (Problem 7.6 in Shanmugan)

Solution:

- Signal model:

$$[X, Y]^T \sim \mathcal{N}\Big(\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}, \begin{bmatrix}\sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2\end{bmatrix}\Big)$$

  with

$$\rho = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X\sigma_Y}$$

  denoting the correlation coefficient of X and Y.

- Conditional expectation $E[Y|X=x]$:

  From (2.66a) in Shanmugan, we obtain

$$\begin{aligned} E[Y|X=x] &= \mu_Y + \underbrace{\Sigma_{YX}}_{\rho\sigma_X\sigma_Y}\underbrace{\Sigma_{XX}^{-1}}_{\sigma_X^{-2}}(x-\mu_X) \\ &= \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X), \\ E[Y|X=x] &= \underbrace{\mu_Y - \rho\frac{\sigma_Y}{\sigma_X}\mu_X}_{a} + \underbrace{\rho\frac{\sigma_Y}{\sigma_X}}_{b}x. \end{aligned}$$

- Mean square error:

  According to the lecture notes, the mean square error can be written as:

$$\begin{aligned} E[(Y-\hat{Y})^2] &= E[(Y-\hat{Y})Y] \\ &= E[Y^2] - E[\hat{Y}Y] \end{aligned}$$

  (See (3.3b) in the lecture notes).

  Therefore,

$$\begin{aligned} E\big\{[Y-(a+bX)]^2\big\} &= E\big\{[Y-(a+bX)]\cdot Y\big\} \\ &= E\big\{[YY-(a+bX)Y]\big\} \\ &= E[Y^2] - aE[Y] - bE[XY] \\ &= E[Y^2] - a\mu_Y - bE[XY]. \end{aligned} \tag{3}$$

  We know from the lecture notes (page 4-1, definition of variance of Y) that

$$E[Y^2] = \mu_Y^2 + \Sigma_{YY},$$

  and

$$\begin{aligned} a &= \mu_Y - b\mu_X, && \text{(3.4b) in the lecture notes} \\ b &= \rho\frac{\sigma_Y}{\sigma_X}. && \text{(3.4a) in the lecture notes} \end{aligned}$$

Inserting in (3) yields

$$\begin{aligned} E\big\{[Y-(a+bX)]^2\big\} &= \sigma_Y^2+\mu_Y^2-\mu_Y^2+b\mu_X\mu_Y-bE[XY] \\ &= \sigma_Y^2+\rho\frac{\sigma_Y}{\sigma_X}\mu_X\mu_Y-\rho\frac{\sigma_Y}{\sigma_X}E[XY] \\ &= \sigma_Y^2-\rho\frac{\sigma_Y}{\sigma_X}\big(E[XY]-\mu_X\mu_Y\big). \end{aligned} \tag{4}$$

According to the definition of Σ_{XY} in the lecture notes (page 4-1), $\Sigma_{XY}=E[XY]-\mu_X\mu_Y$, (4) becomes finally

$$\begin{aligned} E\big\{[Y-(a+bX)]^2\big\} &= \sigma_Y^2-\rho\frac{\sigma_Y}{\sigma_X}E[(X-\mu_X)(Y-\mu_Y)] \\ &= \sigma_Y^2\left[1-\rho\underbrace{\frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X\sigma_Y}}_{=\rho}\right] \\ E\big\{[Y-(a+bX)]^2\big\} &= \sigma_Y^2(1-\rho^2). \end{aligned}$$

Problem 5.3 (Problem 7.8 in Shanmugan)

Solution

Let us consider the two linear estimators:

- the LMMSEE:
$$\hat{Y} = h_0 + \sum_{i=1}^{n} h_i X(i);$$
- an arbitrary linear estimator:
$$\check{Y} = b_0 + \sum_{i=1}^{n} b_i X(i).$$

Then we have to show that

$$E[(Y - \check{Y})^2] \quad \geq \quad E[(Y - \hat{Y})^2]. \tag{5}$$

Let us start with

$$\begin{aligned} E\big[(Y - \check{Y})^2\big] \quad &= \quad E\big\{\big[(Y - \hat{Y}) + (\hat{Y} - \check{Y})\big]^2\big\} \\ &= \quad E\big[(Y - \hat{Y})^2\big] + E\big[(Y - \check{Y})^2\big] + 2E\big[(Y - \hat{Y})(\hat{Y} - \check{Y})\big]. \end{aligned} \tag{6}$$

From the orthogonality principle, we know that

$$E[(Y - \hat{Y})X(i)] = 0, \quad i = 1, ..., n.$$

$\hat{Y}$ and $\check{Y}$ are linear combinations of $X(1), ..., X(n)$, therefore

$$E[(Y - \hat{Y})\hat{Y}] = 0,$$

$$E[(Y - \hat{Y})\check{Y}] = 0.$$

Hence the last term in (6) vanishes:

$$\begin{aligned} E[(Y - \hat{Y})(\hat{Y} - \check{Y})] \quad &= \quad \underbrace{E[(Y - \hat{Y})\hat{Y}]}_{=0} - \underbrace{E[(Y - \hat{Y})\check{Y}]}_{=0} \\ &= \quad 0, \end{aligned} \tag{7}$$

and (5) is proved to be true.