Stochastic Processes II (FP-7.5)
Solution Set 6

Problem 6.1 (Problem 7.15 in Shanmugan)

Solution:

- Orthogonality principle

$$E\big[\big(Y(n)-\hat{Y}(n)\big)X(n-k)\big] \;=\; 0, \quad \forall k=-M,...,+M \tag{1}$$

Substituting (1) from the exercise set in the left-hand of equation (1) above and applying the property of WSS process yields

$$\underbrace{E[Y(n)X(n-k)]}_{\substack{=E[X(n)Y(n+k)]\\ =R_{XY}(k)}} - \sum_{m=-M}^{+M} h(m)\underbrace{E[X(n-k)X(n-m)]}_{\substack{=E[X(n)X(n+k-m)]\\ =R_{XX}(k-m)}} \;=\; 0$$

Hence

$$R_{XY}(k) = \sum_{m=-M}^{+M} h(m) R_{XX}(k-m), \quad k=-M,...,+M$$

We can rewrite the $2M+1$ equations above in a matrix form as

$$\underbrace{\begin{bmatrix} R_{XY}(-M) \\ R_{XY}(-M+1) \\ \vdots \\ R_{XY}(M-1) \\ R_{XY}(M) \end{bmatrix}}_{\boldsymbol{R}_{XY}} = \underbrace{\begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \dots & R_{XX}(2M-1) & R_{XX}(2M) \\ R_{XX}(1) & R_{XX}(0) & \dots & R_{XX}(2M-2) & R_{XX}(2M-1) \\ \vdots & \vdots & & \vdots & \vdots \\ R_{XX}(2M-1) & R_{XX}(2M-2) & \dots & R_{XX}(0) & R_{XX}(1) \\ R_{XX}(2M) & R_{XX}(2M+1) & \dots & R_{XX}(1) & R_{XX}(0) \end{bmatrix}}_{\boldsymbol{R}_{XX}} \cdot \underbrace{\begin{bmatrix} h(-M) \\ h(-M+1) \\ \vdots \\ h(M-1) \\ h(M) \end{bmatrix}}_{\boldsymbol{h}} \tag{2}$$

- Coefficient vector of the LMMSEE:

Provided $\boldsymbol{R}_{XX}$ is invertible,

$$\boldsymbol{h} = \boldsymbol{R}_{XX}^{-1}\boldsymbol{R}_{XY}.$$

Such a LMMSEE is also called a finite Wiener filter.

Problem 6.2

Solution:

a) Find the Wiener-Hopf equations for the coefficients of the noncausal Wiener filter of length $2M+1$ and the causal Wiener filter of length $M+1$ estimating $Y(n)$ based on the observation of $X(n)$.

- Non-causal finite Wiener filter with finite length $2M+1$:

$$\hat{Y}(n) = \sum_{m=-M}^{+M} h(m)X(n-m).$$

We can make use of the result in Problem 7.15 to find the coefficients of the noncausal Wiener filter:

$$\begin{aligned} R_{XY}(k) &= E[X(n)Y(n+k)] \\ &= E[(Y(n)+W(n))Y(n+k)] \\ &= \underbrace{E[Y(n)Y(n+k)]}_{=R_{YY}(k)} + \underbrace{E[W(n)Y(n+k)]}_{=0} \\ &= R_{YY}(k) \end{aligned}$$

$$\begin{aligned} R_{XX}(k) &= E[X(n)X(n+k)] \\ &= E[(Y(n)+W(n))(Y(n+k)+W(n+k))] \\ &= E[Y(n)Y(n+k)] + E[W(n)W(n+k)] \\ &= R_{YY}(k) + R_{WW}(k) \\ &= R_{YY}(k) + \frac{1}{4}\delta(k) \end{aligned}$$

In this case, (2) is given by

$$\begin{bmatrix} 0 \\ \vdots \\ \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{4} & 1 & \frac{1}{4} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{4} & 1 & \frac{1}{4} & \dots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \dots & \frac{1}{4} & 1 & \frac{1}{4} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{4} & 1 \end{bmatrix} \cdot \begin{bmatrix} h(-M) \\ \vdots \\ h(-1) \\ h(0) \\ h(1) \\ \vdots \\ h(M) \end{bmatrix}. \tag{3}$$

Solving (3) yields

$$\begin{bmatrix} h(-M) \\ \vdots \\ h(-1) \\ h(0) \\ h(1) \\ \vdots \\ h(M) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 & 0 & \dots & 0 & 0 \\ \frac{1}{4} & 1 & \frac{1}{4} & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{4} & 1 & \frac{1}{4} & \dots & 0 & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \dots & \frac{1}{4} & 1 & \frac{1}{4} & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{4} & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \\ \vdots \\ 0 \end{bmatrix}.$$

- Causal finite Wiener filter with length of $M + 1$:
  The causal Wiener filter is of the form

$$\hat{Y}_c(n) = \sum_{m=0}^{M} h(m)X(n-m)$$

Applying the orthogonality principle, $[h(0), h(1), \dots, h(M)]^T$ satisfies the linear equation

$$\begin{bmatrix} R_{XY}(0) \\ R_{XY}(1) \\ \vdots \\ R_{XY}(M) \end{bmatrix} = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \dots & R_{XX}(M) \\ R_{XX}(1) & R_{XX}(0) & \dots & R_{XX}(M-1) \\ \vdots & & & \\ R_{XX}(M) & R_{XX}(M-1) & \dots & R_{XX}(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M) \end{bmatrix}.$$

Inserting the values yields

$$\begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 & \dots & 0 & 0 \\ \frac{1}{4} & 1 & \frac{1}{4} & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \\ h(M) \end{bmatrix} \tag{4}$$

Solving (4) yields

$$\begin{bmatrix} h(0) \\ h(1) \\ \vdots \\ h(M-1) \\ h(M) \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 & \dots & 0 & 0 \\ \frac{1}{4} & 1 & \frac{1}{4} & \dots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

b) Calculate the filter coefficients for $M = 1$. Compute the mean-square estimation errors resulting when using both filters.

- Non-causal finite Wiener filter with finite length $2M+1$, $M=1$:
  When $M=1$, (3) reduces to

$$\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} & 0 \\ \frac{1}{4} & 1 & \frac{1}{4} \\ 0 & \frac{1}{4} & 1 \end{bmatrix} \cdot \begin{bmatrix} h(-1) \\ h(0) \\ h(1) \end{bmatrix}.$$

  Solving yields

$$\begin{bmatrix} h(-1) \\ h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} \frac{15}{14} & -\frac{2}{7} & \frac{1}{14} \\ -\frac{2}{7} & \frac{8}{7} & -\frac{2}{7} \\ \frac{1}{14} & -\frac{2}{7} & \frac{15}{14} \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{14} \\ \frac{5}{7} \\ \frac{1}{14} \end{bmatrix}$$

  The mean-square estimation error is calculated to be

$$E\big[\big(Y(n) - \hat{Y}_c(n)\big)^2\big] = R_{YY}(0) - \sum_{m=-M}^{M} h(m) R_{XY}(m).$$

  When $M=1$, the mean-square estimation error reads

$$\begin{aligned} E[(Y(n) - \hat{Y}(n))^2] &= R_{YY}(0) - \sum_{m=-1}^{+1} h(m) R_{XY}(m) \\ &= \frac{3}{4} - [\frac{1}{14} \cdot \frac{1}{4} + \frac{5}{7} \cdot \frac{3}{4} + \frac{1}{14} \cdot \frac{1}{4}] \\ &= \frac{5}{28}. \end{aligned}$$

  Comment:
  Using the same method, we calculated different non-causal Wiener filters with $M = 0, 1, 2, .., 10$. Figure (1) shows that as M increases the mean-square estimation error curve (marked by $*$) converges to a stable level that coincides with the MSE of the non-causal Wiener filter. It can be concluded that the finite Wiener filter with $M=3$ provides a good approximation of the non-causal Wiener filter.

- Causal finite Wiener filter with finite length $2M+1$, $M=1$:
  When $M=1$, (4) reduces to

$$\begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{4} \\ \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \end{bmatrix}.$$

  Thus the coefficient is given by

$$\begin{bmatrix} h(0) \\ h(1) \end{bmatrix} = \begin{bmatrix} \frac{16}{15} & -\frac{4}{15} \\ -\frac{4}{15} & \frac{16}{15} \end{bmatrix} \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{15} \\ \frac{1}{15} \end{bmatrix}.$$

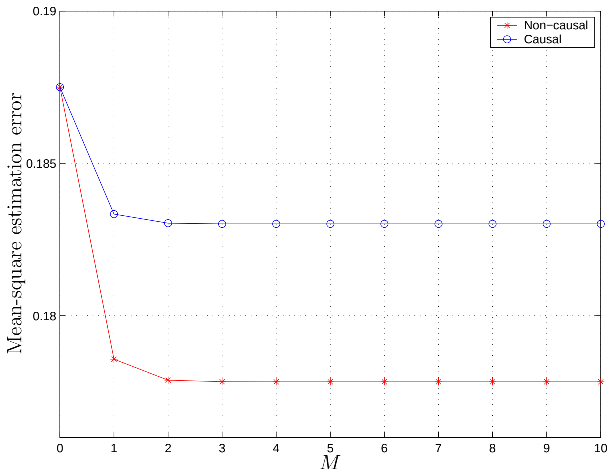

Figure 1: Mean-square estimation error for the non-causal and the causal finite Wiener filters versus M.

The mean-square estimation error is calculated to be

$$E\big[\big(Y(n) - \hat{Y}_c(n)\big)^2\big] \quad = \quad R_{YY}(0) - \sum_{m=0}^{M} h(m)R_{XY}(m).$$

When $M = 1$, the mean-square estimation error is

$$\begin{aligned} E\big[\big(Y(n) - \hat{Y}_c(n)\big)^2\big] &= R_{YY}(0) - \sum_{m=0}^{1} h(m)R_{XY}(m) \\ &= \frac{3}{4} - \Big[\frac{11}{15} \cdot \frac{3}{4} + \frac{1}{15} \cdot \frac{1}{4}\Big] \\ &= \frac{11}{60}. \end{aligned}$$

Comments:

The causal Wiener filters, with $M = 0, 1, 2, \ldots, 10$ are calculated as well. The curve marked with circles in Fig (1) represents the mean-square estimation error M. We may observe similar situation that the mean-square estimation error stabilizes after $M \geq 3$. Hence the optimal estimator that can be achieved is the Wiener filter with length of 4, i.e. $M = 3$.

By the way it can be also observed that the non-causal estimators performed better than the causal ones, in the sense that the mean-square estimation errors caused by the non-causal filters are always smaller than (when $M \geq 1$) or equal (when $M = 0$) to that of causal filters, i.e.,

$$E\big[\big(Y(n) - \hat{Y}(n)\big)^2\big] \leq E\big[\big(Y(n) - \hat{Y}_c(n)\big)^2\big].$$