Stochastic Processes II (FP-7.5)
Solution set 7

Problem 7.1

Solution:

- See the lecture note for derivation. Note that in this problem $a(n) = 1$, $h(n) = 0.5$, $\sigma_z^2 = 2$.

The result is summarized as follow:

-Prediction step:

$$\begin{aligned} \hat{Y}(n+1|n) &= h(n+1)\hat{Y}(n|n) \\ &= 0.5\hat{Y}(n|n) \\ R(n+1|n) &= h(n+1)^2 R(n|n) + \sigma_z^2(n+1) \\ &= 0.25R(n|n) + 2 \end{aligned} \tag{1}$$

-Updating step:

$$\hat{Y}(n+1|n+1) = \hat{Y}(n+1|n) + b(n+1)[X(n+1) - \hat{X}(n+1|n)] \tag{2}$$

$$\begin{aligned} R(n+1|n+1) &= [1 - b(n+1)a(n+1)]R(n+1|n) \\ &= [1 - b(n+1)]R(n+1|n) \end{aligned} \tag{3}$$

with

$$\begin{aligned} b(n+1) &= \frac{a(n+1)R(n+1|n)}{a(n+1)^2 R(n+1|n) + \sigma_w^2(n+1)} \\ &= \frac{R(n+1|n)}{R(n+1|n) + \sigma_w^2(n+1)} \end{aligned}$$

- The blockdiagram of the Kalman filter is shown in figure 1.

- Calculate the estimate $\hat{Y}(n|n)$ for n=1,2:

  Initialize the filter:

$$\begin{aligned} \hat{Y}(0|0) &= E[Y(0)] = 0 \\ R(0|0) &= E[Y^2(0)] = 1 \end{aligned} \tag{4}$$

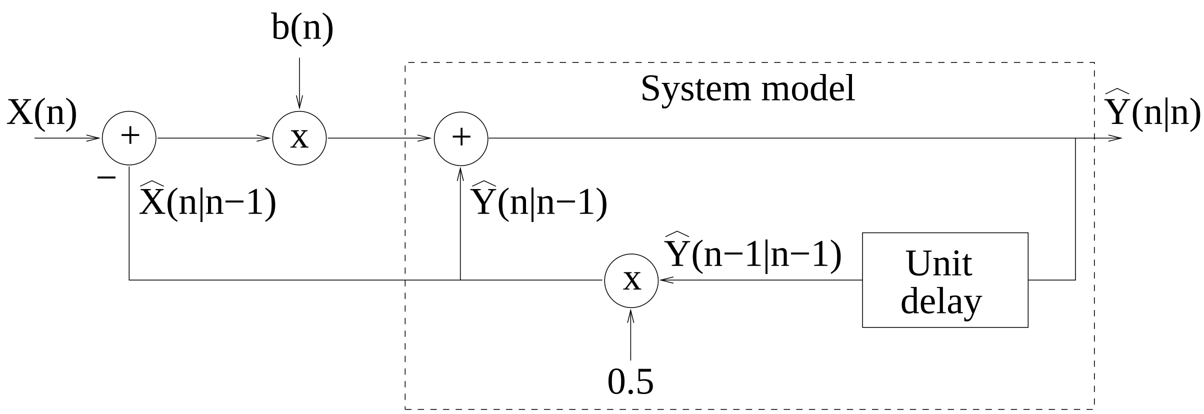

Figure 1: The block diagram of Kalman filter

First predict the $\hat{Y}(1)$,

$$\begin{aligned} \hat{Y}(1|0) &= h(1)\hat{Y}(0|0) \\ &= \frac{1}{2}(0) = 0 \\ R(1|0) &= h(1)^2 R(0|0) + \sigma_z^2(1) = \frac{1}{4}(1) + 2 = 2.25 \end{aligned} \tag{5}$$

Next, as $X(1)$ is observed, we correct the predicted estimate,

$$\begin{aligned} b(1) &= \frac{R(1|0)}{\sigma_w^2(1) + R(1|0)} \\ &= \frac{\frac{9}{4}}{1 + \frac{9}{4}} = \frac{9}{13} \\ \hat{Y}(1|1) &= \hat{Y}(1|0) + b(1)(X(1) - \hat{Y}(1|0)) \\ &= \frac{9}{13}X(1) \end{aligned} \tag{6}$$

Update the minimum MSE:

$$\begin{aligned} R(1|1) &= (1 - b(1))R(1|0) \\ &= (1 - \frac{9}{13})\frac{9}{4} = \frac{9}{13} \end{aligned} \tag{7}$$

When n=2, the result is

$$
\begin{aligned}
\hat{Y}(2|1) &= \frac{9}{26}X(1) \\
R(2|1) &= \frac{113}{52} \\
b(2) &= \frac{113}{139} \\
\hat{Y}(2|2) &= \frac{9}{26}X(1) + \frac{113}{139}(X(2) - \frac{9}{26}X(1)) = \frac{9}{139}X(1) + \frac{113}{139}X(2) \\
R(2|2) &= \frac{113}{238}
\end{aligned} \tag{8}
$$

- In figure 2, the minimum MSE is plotted as a function of n.

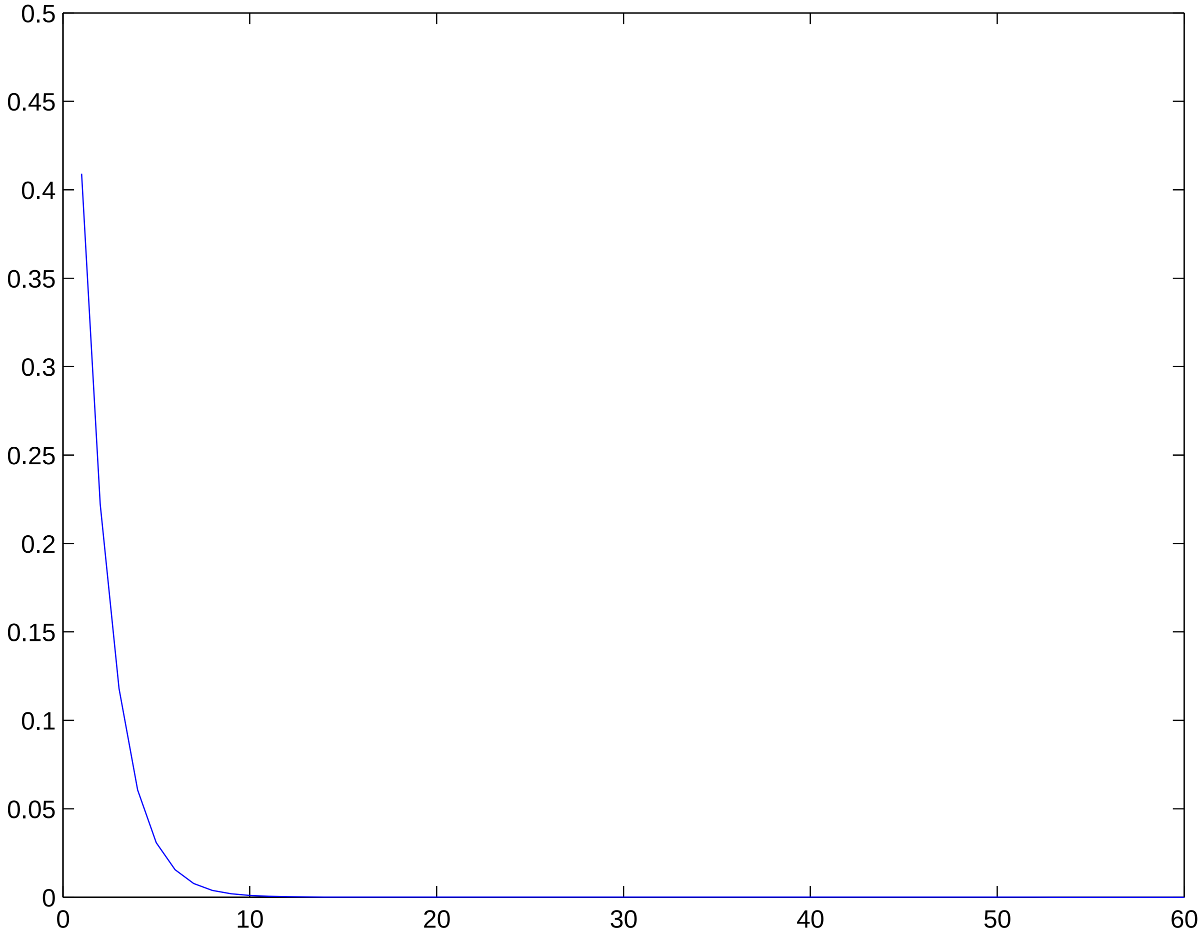

Figure 2: