Stochastic Processes II (FP-7.5)
Solution Set 8

Problem 8.1 (Example 7.17 in Shanmugan)

Solution:

In Example 7.17, it is assumed that $S_1(t)$ represents the position of a particle, and $S_2(t)$ represents its velocity. Then we can write

$$S_1(t) = \int_0^t S_2(\tau)d\tau + S_1(0)$$

or with a sampling time of one second, this can be converted into the approximate difference equation

$$S_1(n) = S_2(n-1) + S_1(n-1). \tag{1}$$

We assume that the sequence

$$W(n) = S_2(n) - S_2(n-1) \tag{2}$$

of differences between consecutive velocity samples is a zero-mean white noise sequence with unity variance, i.e.

$$\begin{aligned} E[W(n)] &= 0, \\ E[W(n)W(n+k)] &= \delta(k). \end{aligned}$$

Equations (1) and (2) can be combined in the matrix notation (system model)

$$\boldsymbol{Y}(n) = \boldsymbol{H}(n)\boldsymbol{Y}(n-1) + \boldsymbol{Z}(n), \quad n = 1, 2, 3, \ldots$$

where

- $\boldsymbol{Y}(n) \doteq \begin{bmatrix} S_1(n) & S_2(n) \end{bmatrix}^T$;
- $\boldsymbol{H}(n) \doteq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$;
- $\boldsymbol{Z}(n) \doteq \begin{bmatrix} 0 \\ W(n) \end{bmatrix}$.

From the above assumption on $W(n)$, the covariance matrix of $\boldsymbol{Z}(n)$ reads

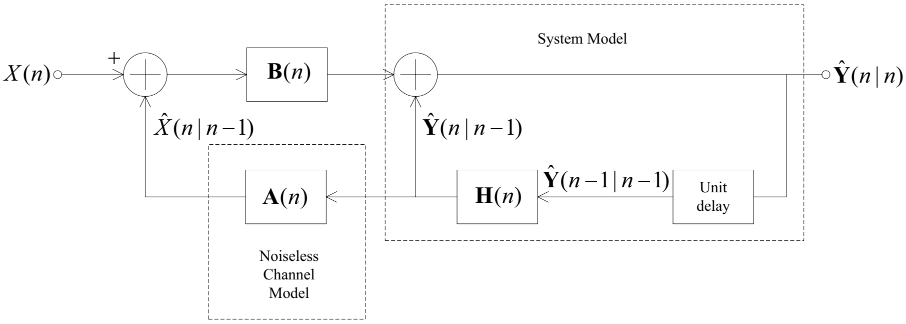
$$\begin{aligned} \boldsymbol{Q}_Z(n) &= E[\boldsymbol{Z}(n)\boldsymbol{Z}(n)^T] \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The channel model can be written as

$$X(n) = \boldsymbol{A}(n)\boldsymbol{Y}(n) + N(n)$$

where

- $X(n)$ is a scalar, representing the observed position of the particle;
- $\boldsymbol{A}(n) = \begin{bmatrix} 1 & 0 \end{bmatrix}$;
- $N(n)$ is a zero-mean white noise sequence with unit variance: $E[N(n)] = 0$ and $E\big[N(n)N(n+k)\big] = \delta(k)$, i.e. $Q_N(n) = 1$.



Assuming the initial state $\boldsymbol{Y}(0|0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\boldsymbol{R}(0|0) = \sigma^2 \boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, we solve:

- Prediction $(n = 1)$

$$\begin{aligned} \hat{\boldsymbol{Y}}(1|0) &= \boldsymbol{H}(1)\boldsymbol{Y}(0|0) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \hat{X}(1|0) &= \boldsymbol{A}(1)\hat{\boldsymbol{Y}}(1|0) \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \boldsymbol{R}(1|0) &= \boldsymbol{H}(1)\boldsymbol{R}(0|0)\boldsymbol{H}(1)^T + \boldsymbol{Q}_Z(1) \\ &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

- Update ($n=1$)

$$\begin{aligned}
\boldsymbol{B}(1) &= \boldsymbol{R}(1|0)\boldsymbol{A}(1)^T[\boldsymbol{A}(1)\boldsymbol{R}(1|0)\boldsymbol{A}(1)^T + Q_N(1)]^{-1} \\
&= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \right]^{-1} \\
&= \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{Y}}(1|1) &= \hat{\boldsymbol{Y}}(1|0) + \boldsymbol{B}(1) \cdot [X(1) - \hat{X}(1|0)] \\
&= \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \cdot [X(1) - 0] \\
&= \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} X(1)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{R}(1|1) &= [\boldsymbol{I} - \boldsymbol{B}(1)\boldsymbol{A}(1)]\boldsymbol{R}(1|0) \\
&= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right] \cdot \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{3} \end{bmatrix}
\end{aligned}$$

- Prediction ($n=2$)

$$\begin{aligned}
\hat{\boldsymbol{Y}}(2|1) &= \boldsymbol{H}(2)\hat{\boldsymbol{Y}}(1|1) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} X(1) \\
&= \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} X(1)
\end{aligned}$$

$$\begin{aligned}
\hat{X}(2|1) &= \boldsymbol{A}(2)\hat{\boldsymbol{Y}}(2|1) \\
&= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{3} \end{bmatrix} X(1) \\
&= X(1)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{R}(2|1) &= \boldsymbol{H}(2)\boldsymbol{R}(1|1)\boldsymbol{H}(2)^T + \boldsymbol{Q}_Z(2) \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 2 \\ 2 & \frac{8}{3} \end{bmatrix}
\end{aligned}$$

- Update ($n = 2$)

$$\begin{aligned}
\boldsymbol{B}(2) &= \boldsymbol{R}(2|1)\boldsymbol{A}(2)^T[\boldsymbol{A}(2)\boldsymbol{R}(2|1)\boldsymbol{A}(2)^T + Q_N(2)]^{-1} \\
&= \begin{bmatrix} 3 & 2 \\ 2 & \frac{8}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left[\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 2 & \frac{8}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \right]^{-1} \\
&= \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{Y}}(2|2) &= \hat{\boldsymbol{Y}}(2|1) + \boldsymbol{B}(2) \cdot [X(2) - \hat{X}(2|1)] \\
&= \begin{bmatrix} X(1) \\ \frac{1}{3}X(1) \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} \cdot [X(2) - X(1)] \\
&= \begin{bmatrix} \frac{3}{4}X(2) + \frac{1}{4}X(1) \\ \frac{1}{2}X(2) - \frac{1}{6}X(1) \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{R}(2|2) &= [\boldsymbol{I} - \boldsymbol{B}(2)\boldsymbol{A}(2)]\boldsymbol{R}(2|1) \\
&= \left[\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right] \cdot \begin{bmatrix} 3 & 2 \\ 2 & \frac{8}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{5}{3} \end{bmatrix}
\end{aligned}$$

Problem 8.2

Solution:

Implement the prediction and update equations from the lecture notes. This is done in the file kalman.m.