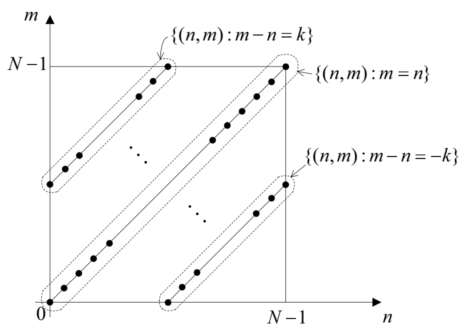
Stochastic Processes II(FP-7.5) Solution Set 9

Solution 9.1

$$
\begin{aligned}
\mathrm{Var}(\overline{X}) &= E\big[(\overline{X}-\mu_{\overline{X}})^2\big] \\
&= E\big[(\overline{X}-\mu_X)^2\big] \qquad (\mu_{\overline{X}}=\mu_X) \\
&= E\left[\left(\frac{1}{N}\sum_{n=0}^{N-1}X_n-\mu_X\right)^2\right] \\
&= E\left[\left(\frac{1}{N}\sum_{n=0}^{N-1}(X_n-\mu_X)\right)^2\right] \\
&= \frac{1}{N^2}E\left[\left(\sum_{n=0}^{N-1}(X_n-\mu_X)\right)^2\right] \\
&= \frac{1}{N^2}E\left[\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}(X_n-\mu_X)(X_m-\mu_X)\right] \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\underbrace{E\big[(X_n-\mu_X)(X_m-\mu_X)\big]}_{C_{XX}(m-n)} \\
&= \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}C_{XX}(m-n) \\
&= \frac{1}{N^2}\sum_{k=-(N-1)}^{N-1}(N-|k|)C_{XX}(k)
\end{aligned}
$$



$$
\mathrm{var}(\overline{X}) = \frac{1}{N}\sum_{k=-(N-1)}^{N-1}\left(1-\frac{|k|}{N}\right)C_{XX}(k)
$$

Solution 9.2

According to the lecture notes (page 4), the sample autocorrelation function $\hat{R}_{XX}(k)$ reads

$$\hat{R}_{XX}(k) = \frac{1}{N} \sum_{n=-\infty}^{+\infty} X_{\text{obs}}(n) X_{\text{obs}}(n+k)$$

Since $X_{obs}(n)$ is real we have symmetry; $\hat{R}_{XX}(k) = \hat{R}_{XX}(-k)$, i.e.

$$\hat{R}_{XX}(k) = \frac{1}{N} \sum_{n=-\infty}^{+\infty} X_{\text{obs}}(n) X_{\text{obs}}(n-k)$$

Let

$$z(k) = \sum_{n=-\infty}^{+\infty} X_{\text{obs}}(n) X_{\text{obs}}(n-k).$$

$$z(k) = \sum_{n=-\infty}^{+\infty} X_{\text{obs}}(n) X_{\text{obs}}(-(k-n)).$$

The above expression can be recast as

$$z(k) = X_{\text{obs}}(k) * X_{\text{obs}}(-k).$$

Applying Fourier transformation on both sides yields

$$\begin{aligned} Z(f) &= X(f) \cdot X(f)^* \\ &= |X(f)|^2. \end{aligned}$$

Hence

$$\begin{aligned} \hat{S}_{XX}(f) &= \mathcal{F}\{\frac{1}{N} z(k)\} \\ &= \frac{1}{N} |X(f)|^2. \end{aligned}$$

Solution 9.3

The Sunspot Numbers data file and the Matlab routine can be downloaded from the course webpage.

- The sample autocorrelation function and the sample autocovariance function: fig(1).

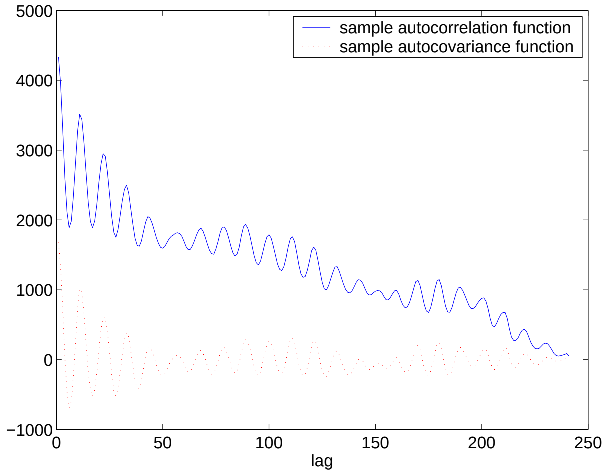


Figure 1:

- The PSD by Periodogram: fig(2). There is a peak at frequency 0.0104 cycles/year,

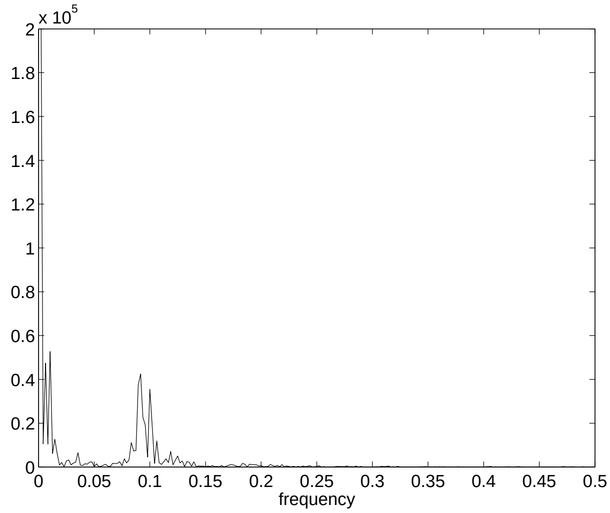


Figure 2:

and a peak at 0.0917 cycles/year, corresponding to periods of 96.15 years and 10.9 years respectively.

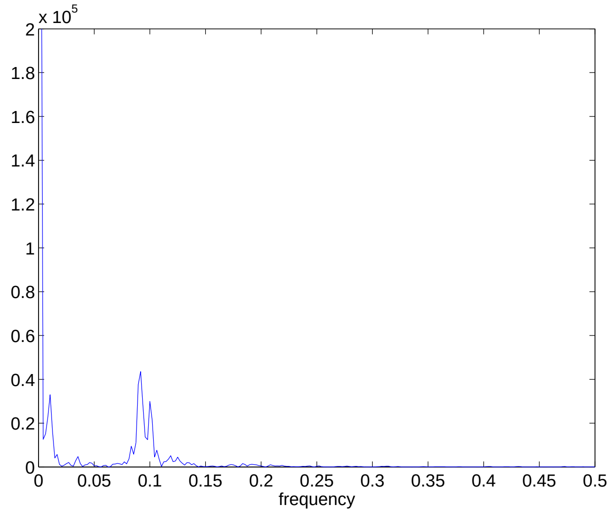


Figure 3:

- The PSD by Blackman-Tukey: fig(3). There is peaks at 0.0917 cycles/year and 0.0104 cycles/year, corresponding to periods of 10.9 years and 96.15 years respectively.