Stochastic Processes II (FP-7.5):

Written Examination

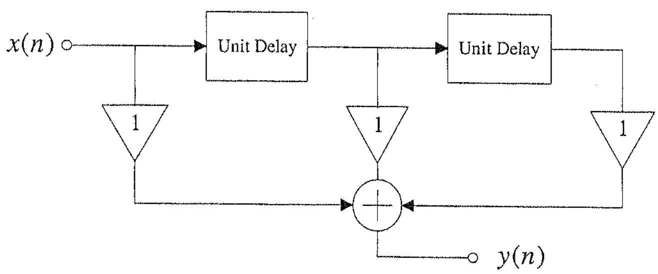
Date and Time: Friday, Feb. 13, 2004, 11.00-14.00.

Important remark: This document contains **4 pages.** Please check that it is complete.

Hints: Try to solve as many questions in the problems as you can. All questions have the same weight. Thus, should you encounter some difficulty in solving one question, then skip it and go to the next.

Problem 1: Linear systems and WSS processes

Let us consider the transversal filter (TF) depicted below,



1.1. Write the equation that relates the output sequence $\{y(n)\}$ to the input sequence $\{x(n)\}$ (input-output relationship of the TF).

1.2. Derive and plot the impulse response $h(n)$ of the TF.

1.3. Show that the transfer function of the TF is $H(f)=\frac{\sin(3\pi f)}{\sin(\pi f)}\cdot\exp\{-j2\pi f\}$.

Hints: (1) $\exp\{jn2\pi f\}=[\exp\{j2\pi f\}]^n$;

(2) $\sum_{n=0}^{N} a^n = \frac{1-a^{(N+1)}}{1-a}$.

1.4. Compute and plot the amplitude spectrum $|H(f)|$ of the TF.

Hint: Using L'Hospital's Rule, we have $\lim_{f\to 0}\frac{\sin(3\pi f)}{\sin(\pi f)}=3$.

1.5. Compute and plot the autocorrelation function $R_{hh}(k)=\sum_n h(n)h(n+k)$ of the impulse response $h(n)$.

1.6. Compute and plot the Fourier transform of $R_{hh}(k)$.

Let us now assume that the input sequence is a zero-mean WSS process $\{X(n)\}$ with autocorrelation function,

$$R_{XX}(k)=\mathbf{E}[X(n)X(n+k)]=\begin{cases}2 & ; \quad k=0\\ 0 & ; \quad k\neq 0\end{cases}$$

1.7. Compute the variance of $\{X(n)\}$.

1.8. Compute and plot the power spectrum $S_{XX}(f)$ of $\{X(n)\}$.

1.9. Compute the integral of $S_{XX}(f)$ over the interval $[-0.5, 0.5]$.

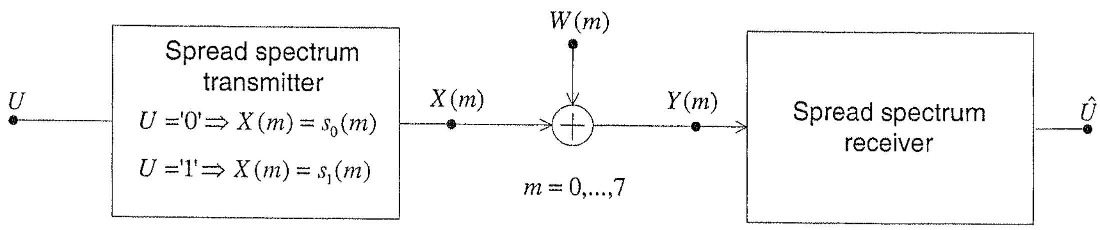
1.10. Derive the expectation $\mathbf{E}[Y(n)]$ and the autocorrelation function $R_{YY}(k) = \mathbf{E}[Y(n)Y(n+k)]$ of the output process $\{Y(n)\}$.
Plot the autocorrelation function $R_{YY}(k)$.

1.11. Derive and plot the power spectrum $S_{YY}(f)$ of $\{Y(n)\}$.

1.12. Compute the variance of $\{Y(n)\}$.

Problem 2: Detection of spread-spectrum signals corrupted by additive white Gaussian noise

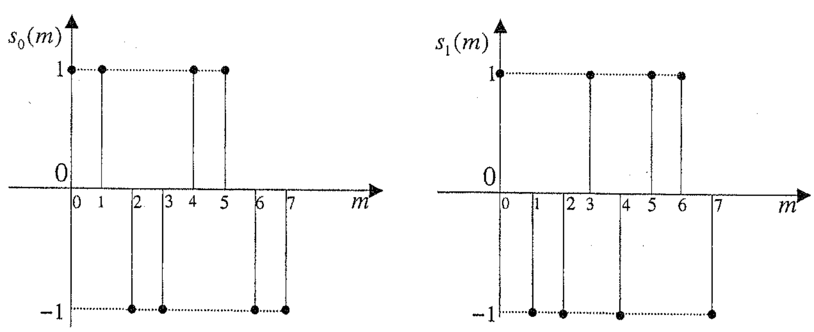
Let us consider the following discrete-time model of a spread-spectrum (SS) communication system using Walsh codes, which transmits across an additive white Gaussian noise channel.



The signal at the transmitter output depends on the input bit U as follows:

$$X(m) = \begin{cases} s_0(m); & U = '0' \\ s_1(m); & U = '1' \end{cases}, \quad m = 0,\ldots,7 .$$

The signals transmitted under both hypotheses are depicted below:



The received signal reads

$$Y(m) = X(m) + W(m), \quad m = 0,\ldots,7.$$

We define the 7-dimensional vectors:

$$\mathbf{X} = [X(0), X(1), \ldots, X(7)]^{\mathrm{T}},$$
$$\mathbf{W} = [W(0), W(1), \ldots, W(7)]^{\mathrm{T}},$$
$$\mathbf{Y} = [Y(0), Y(1), \ldots, Y(7)]^{\mathrm{T}},$$
$$\mathbf{s}_0 = [s_0(0), s_0(1), \ldots, s_0(7)]^{\mathrm{T}},$$
$$\mathbf{s}_1 = [s_1(0), s_1(1), \ldots, s_1(7)]^{\mathrm{T}},$$

where $[\cdot]^{\mathrm{T}}$ denotes the transpose operator.

The vector $\mathbf{W}$ is a white Gaussian noise vector with component variance $\sigma_W^2 = \mathbf{E}[W(m)^2], \quad m = 0, \ldots, 7$.

2.1. Show that the signal vectors $\mathbf{s}_0$ and $\mathbf{s}_1$ have energy $E_s = E_{s_0} = E_{s_1} = 8$.
Thus the signal to noise ratio (SNR) $\eta = \frac{E_s}{\sigma_W^2}$ is equal to $\eta = \frac{8}{\sigma_W^2}$.
In the sequel we assume that $\eta = 4$.

2.2. Show that the scalar product $\mathbf{s}_0{}^{\mathrm{T}}\mathbf{s}_1$ between $\mathbf{s}_0$ and $\mathbf{s}_1$ is equal to 0.

2.3. Formulate the detection process in the SS receiver as a binary hypothesis testing problem where $\mathbf{Y}$ is the observation vector and the null hypothesis is $H_0 \equiv [\text{"}U = \text{'0'"}]$, i.e. that the bit at the input of the SS transmitter is a '0'.

2.4. Find the probability density function of $\mathbf{Y}$ under both hypotheses, i.e. $f(\mathbf{y} \mid H_0)$ and $f(\mathbf{y} \mid H_1)$.

2.5. Calculate the log-likelihood ratio $\ell(\mathbf{y}) = \ln\left(\frac{f(\mathbf{y} \mid H_1)}{f(\mathbf{y} \mid H_0)}\right)$.

2.6. Find the MAP decision rules for the two following a priori probability distributions:

a) $\mathbf{P}[H_0] = \mathbf{P}[H_1] = \frac{1}{2}$,

b) $\mathbf{P}[H_0] = \frac{\exp(3/2)}{1+\exp(3/2)} \approx 0.8176$, $\mathbf{P}[H_1] = \frac{1}{1+\exp(3/2)} \approx 0.1824$.

2.7. Show that the probability density functions of the random variable $Z = \mathbf{Y}^{\mathrm{T}}(\mathbf{s}_1 - \mathbf{s}_0)$ under both hypotheses are given by

$$f(z \mid H_0) = \frac{1}{\sqrt{2\pi} \cdot 4\sigma_W} \exp\left\{-\frac{1}{32\sigma_W^2}(z+8)^2\right\},$$
$$f(z \mid H_1) = \frac{1}{\sqrt{2\pi} \cdot 4\sigma_W} \exp\left\{-\frac{1}{32\sigma_W^2}(z-8)^2\right\}.$$

Hint: Make use of the result of Item 2.2.
Comment: Notice that the decision rules in Item 2.6 can be expressed as $Z \underset{H_0}{\overset{H_1}{\gtrless}} \gamma_{MAP}$, where the threshold γ_{MAP} depends on $\mathbf{P}[H_0]$ and $\mathbf{P}[H_1]$.

2.8. Sketch the areas under the graphs of $f(z \mid H_0)$ and $f(z \mid H_1)$ that correspond to the false alarm probability P_f and the probability of a miss P_m, respectively, for the two MAP rules specified in Item 2.6.

2.9. Calculate P_f and P_m for the two MAP rules specified in Item 2.6.

Hint: Make use of Table D.1 in Appendix D, p.631 of [Shanmugan] or the subsequent table.

2.10. Compute the probability P_e of making a decision error (bit-error probability) for the two MAP rules specified in Item 2.6.

2.11. Compute P_e for the first MAP rule specified in Item 2.6 (a) when in fact the a-priori probability distribution of H_0 and H_1 coincides with that given in Item 2.6 (b). Compare with the results obtained in Item 2.10.

y	$Q(y)$
0.1	4.602×10^{-1}
0.2	4.207×10^{-1}
0.3	3.821×10^{-1}
0.4	3.446×10^{-1}
0.5	3.085×10^{-1}
0.6	2.743×10^{-1}
0.7	2.420×10^{-1}
0.8	2.119×10^{-1}
0.9	1.841×10^{-1}
1.0	1.587×10^{-1}
1.1	1.357×10^{-1}
1.2	1.151×10^{-1}
1.3	0.968×10^{-1}
1.4	0.808×10^{-1}
1.5	0.668×10^{-1}
1.6	0.548×10^{-1}
1.7	0.446×10^{-1}
1.8	0.359×10^{-1}
1.9	0.287×10^{-1}
2.0	0.228×10^{-1}