Stochastic Processes II (FP-7.5):

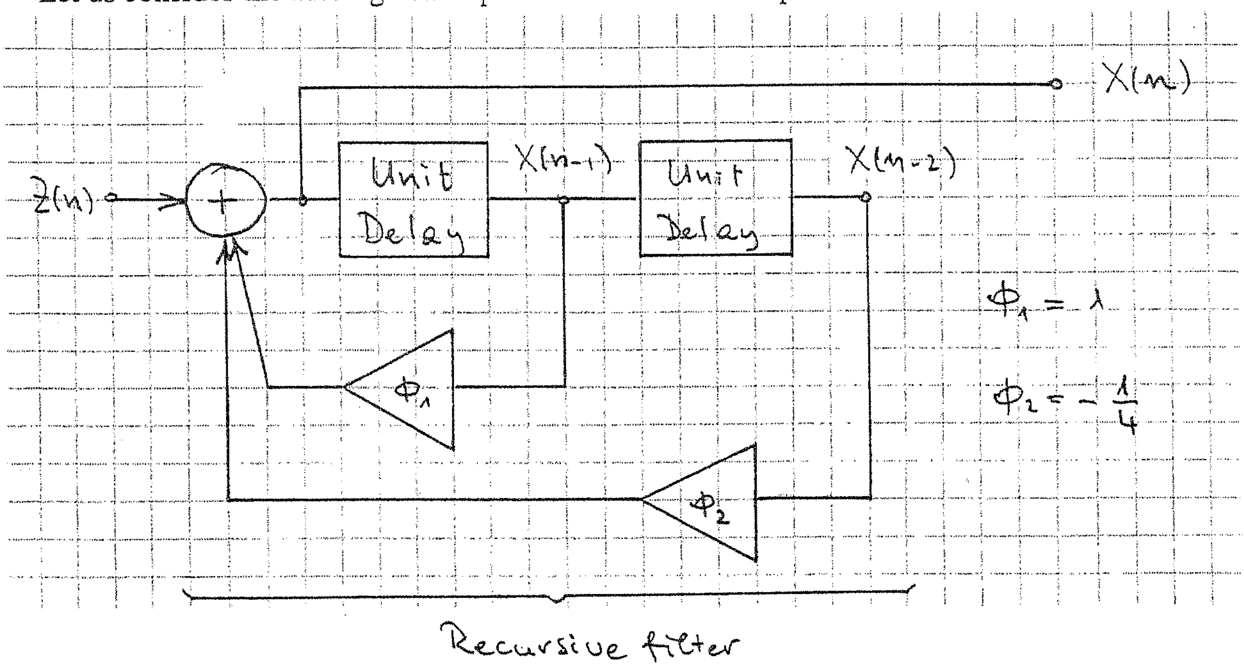
Written Examination

Date and Time: Monday, Jan. 15, 2001, 09.00-11.00.

Hints: Try to solve as many items in the problems as you can. All items have the same weight. Thus, should you encounter some difficulty in deriving a solution, then skip the question and go to the next.

Problem 1: AR(2) process

Let us consider the autoregressive process of order two depicted below:



The input sequence is a white noise sequence $\{Z(n)\}$ with unit variance, i.e. $E[Z(n)] = 0$ and $E[Z(n)Z(n+k)] = \sigma_Z^2 \delta(k)$ with $\sigma_Z^2 = 1$.

1.1. Write the equation that relates the output sequence $\{X(n)\}$ to the input sequence $\{Z(n)\}$ (input-output relationship of the recursive filter).

1.2. Compute the values of the impulse response $h(n)$ of the recursive filter for $n = 0, 1, \ldots, 5$.

1.3. Derive the transfer function $H(f)$ of the recursive filter. *Hint:* Remember that $\exp(-j4\pi f) = [\exp(-j2\pi f)]^2$.

1.4. Derive the amplitude spectrum $|H(f)|$ of the recursive filter and plot its graph.

1.5. Derive the power spectrum $S_{XX}(f)$ of $\{X(n)\}$ and sketch its graph.

1.6. Make use of the Yule-Walker equations to compute the values of the auto-correlation function $R_{XX}(k) = \mathrm{E}[X(n)X(n+k)]$ of $\{X(n)\}$ for $k = 0,1,2$. Derive the result first for arbitrary ϕ_1 and ϕ_2, then insert the selected values for these parameters.

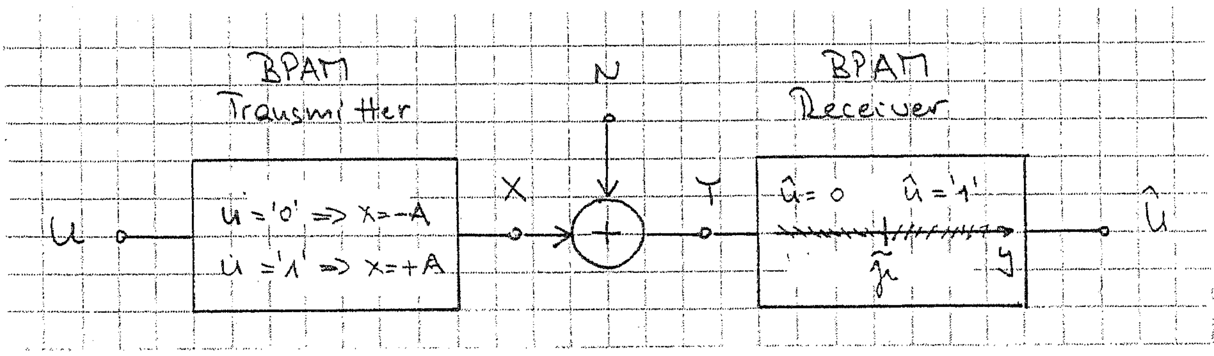
1.7. Write the recursive equation that expresses $R_{XX}(k)$ as a function of $R_{XX}(k-1)$ and $R_{XX}(k-2)$ for $k \geq 1$ first for arbitrary ϕ_1 and ϕ_2, and then for the selected values for these parameters.

1.8. Compute the variance σ_X^2 of $X(n)$.

1.9. What can you say about the integral $\int_{-0.5}^{+0.5} S_{XX}(f)df$?

Problem 2: Detection of BPAM signals corrupted by additive white Gaussian noise

Let us consider the simplified model of a BPAM (Binary Amplitude Modulation) system transmitting across an additive Gaussian channel.



The output level X of the transmitter depends as follows from the input bit U:

$$X = \begin{cases} A & ;U = '1' \\ -A & ;U = '0' \end{cases}, \qquad \text{with } A > 0.$$

The received signals Y reads

$$Y = X + N,$$

where N is a zero-mean Gaussian random variable with variance $\sigma_N^2 = E[N^2]$. The signal-to-noise ratio $\eta = \frac{E[X^2]}{E[N^2]} = \left(\frac{A}{\sigma_N}\right)^2$ is supposed to equal 10 dB, i.e. $\eta = 10$.

2.1. Formulate the detection process of the received BPAM signal as a binary hypothesis testing problem where Y is the observation and the null hypothesis is $H_0 \equiv ["U = '1'"]$, i.e. corresponds to a '1' at the BPAM transmitter input.

2.2. Find the probability density function of Y under both hypotheses, i.e. $f(y \mid H_0)$ and $f(y \mid H_1)$.

2.3. Calculate the log-likelihood ratio $\ell(y) = \ln\left(\frac{f(y \mid H_1)}{f(y \mid H_0)}\right)$.

2.4. Find the MAP decision rules for the two following a priori probability distributions:

a) $\mathbf{P}[H_0] = \mathbf{P}[H_1] = \frac{1}{2}$,

b) $\mathbf{P}[H_0] = \frac{1}{1+\exp(5)} \approx 0.0067$, $\mathbf{P}[H_1] = \frac{\exp(5)}{1+\exp(5)} \approx 0.9933$.

Sketch the corresponding decision regions.

2.5. Sketch the areas under the probability densities $f(y \mid H_0)$ and $f(y \mid H_1)$ that correspond to the false alarm probability P_f and the probability of a miss P_m for the two MAP rules specified in 2.4.

2.6. Calculate P_f and P_m for the two MAP rules specified in 2.4.

Hint: Make use of Table D.1 in Appendix D, p. 631 of [Shanmugan].

2.7. Compute the probability P_e of making a decision error (bit-error probability) for the two MAP rules specified in 2.4.

2.8. Compute the probability of making a decision error for the first MAP rule specified in 2.4. when in fact the a-priori probability distribution coincides with that given in 2.4.b). Compare with the results obtained in 2.7.

Solutions of the

2001 Examen in Stochastic Processes II

Problem 1: AR(2) process

1.1. Input - output relationship

$$X(n) = Z(n) + \phi_1 X(n-1) + \phi_2 X(n-2) \qquad (1.1)$$

with the values $\phi_1 = 1$, $\phi_2 = -\frac{1}{4}$:

$$\underline{X(n) = Z(n) + X(n-1) - \frac{1}{4} X(n-2)}$$

1.3 Fourier transform of (1.1)

$$X(f) - \phi_1 X(f) \exp(-j2\pi f) - \phi_2 X(f) \exp(-j4\pi f) = Z(f)$$

$$X(f) \left[1 - \phi_1 \exp(-j2\pi f) - \phi_2 \exp(-j4\pi f)\right] = Z(f)$$

$$H(f) = \frac{X(f)}{Z(f)} = \left[1 - \phi_1 \exp(-j2\pi f) - \phi_2 \exp(-j4\pi f)\right]^{-1}$$

$$= \left[1 - \exp(-j2\pi f) + \frac{1}{4} \exp(-j4\pi f)\right]^{-1}$$

$$= \left[1 - \exp(-j2\pi f) + \left(\frac{1}{2} \exp(-j2\pi f)\right)^2\right]^{-1}$$

$$\underline{H(f) = \left[1 - \frac{1}{2} \exp(-j2\pi f)\right]^{-2}}$$

1.4. $|H(f)| = |1 - \frac{1}{2}\exp(-j2\pi f)|^{-2}$

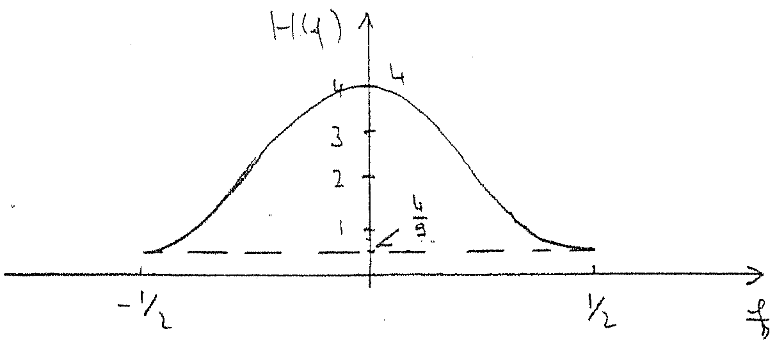
$$= \left[\left(1 - \frac{1}{2}\exp(-j2\pi f)\right)\left(1 - \frac{1}{2}\exp(j2\pi f)\right)\right]^{-1}$$

$$= \left[1 - \frac{1}{2}\exp(+j2\pi f) - \frac{1}{2}\exp(-j2\pi f) + \frac{1}{4}\right]^{-1}$$

$$\underline{|H(f)| = \left|\frac{5}{4} - \cos(2\pi f)\right|^{-1}}$$

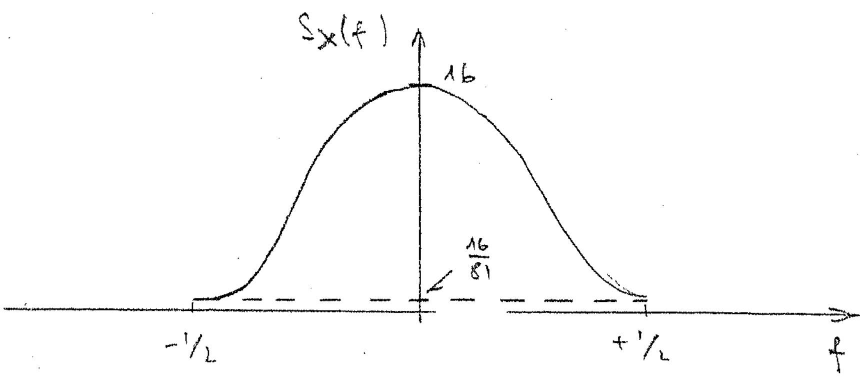
$f=0: \quad |H(0)| = \left|\frac{5}{4} - 1\right|^{-1} = 4$

$f = \frac{1}{2}: \quad |H(\frac{1}{2})| = \left|\frac{5}{4} + 1\right| = \frac{4}{9}$



1.5. $S_x(f) = |H(f)|^2 \cdot S_z(f)$

$$\underline{S_x(f) = \left|\frac{5}{4} - \cos(2\pi f)\right|^{-2}}$$



1.6. Yule - Walker equations (see 2-6 of the lecture notes) :

$$R_{xx}(0) = \left[R_{xx}(1) \ R_{xx}(2)\right]\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} + \sigma_z^2 \qquad (1.2)$$

$$\begin{bmatrix} R_{xx}(1) \\ R_{xx}(2) \end{bmatrix} = \begin{bmatrix} R_{xx}(0) & R_{xx}(1) \\ R_{xx}(1) & R_{xx}(0) \end{bmatrix}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} \qquad \begin{matrix}(1.3)\\(1.4)\end{matrix}$$

From (1.3) :

$$R_{xx}(1) = \phi_1 R_{xx}(0) + \phi_2 R_{xx}(1)$$

$$R_{xx}(1)\left[1 - \phi_2\right] = \phi_1 R_{xx}(0)$$

$$R_{xx}(1) = \frac{\phi_1}{1-\phi_2} R_{xx}(0) \qquad (1.5)$$

From (1.4)

$$R_{xx}(2) = \phi_1 R_{xx}(1) + \phi_2 R_{xx}(0)$$

$$\overset{1.5}{=} \frac{\phi_1^2}{1-\phi_2} R_{xx}(0) + \phi_2 R_{xx}(0)$$

$$= \left[\frac{\phi_1^2}{1-\phi_2} + \phi_2\right] R_{xx}(0) \qquad (1.6)$$

From (1.2)

$$R_{xx}(0) = \phi_1 R_{xx}(1) + \phi_2 R_{xx}(2) + \sigma_z^2$$

$$\overset{(1.5)+(1.6)}{=} \frac{\phi_1^2}{1-\phi_2} R_{xx}(0) + \phi_2\left[\frac{\phi_1^2}{1-\phi_2} + \phi_2\right] R_{xx}(0) + \sigma_z^2$$

Solving yields:

$$R_{xx}(0)\Bigg(\underbrace{1 - \frac{\phi_1^2}{1-\phi_2} - \phi_2\left[\frac{\phi_1^2+\phi_2-\phi_2^2}{1-\phi_2}\right]}\Bigg) = \sigma_z^2$$

$$= \frac{1}{1-\phi_2}\left(1-\phi_2-\phi_1^2-\phi_2\phi_1^2-\phi_2^2+\phi_2^3\right)$$

$$\underline{R_{xx}(0) = \frac{1-\phi_2}{1-\phi_2-\phi_2^2+\phi_2^3-\phi_1^2-\phi_2\phi_1^2}\,\sigma_z^2}$$

Inserting $\phi_1 = 1$, $\phi_2 = -\frac{1}{4}$ yields

$$1-\phi_2 = 1+\frac{1}{4} = \frac{5}{4}$$

$$1-\phi_2-\phi_2^2+\phi_3^2-\phi_1^2-\phi_2\phi_1^2 =$$

$$= \cancel{1}+\frac{1}{4}-\frac{1}{16}-\frac{1}{64}-\cancel{1}+\frac{1}{4} = \frac{27}{64}$$

$$\underline{R_{xx}(0)} = \frac{5/4}{27/64} = \frac{5}{\cancel{4}}\cdot\frac{\overset{16}{\cancel{64}}}{27} = \underline{\frac{80}{27}}$$

$$\underline{R_{xx}(1)} \overset{(1.5)}{=} \frac{4}{5}R_{yy}(0) = \frac{4}{\cancel{5}}\cdot\frac{\overset{16}{\cancel{80}}}{27} = \underline{\frac{64}{27}}$$

$$R_{xx}(2) \overset{(1.6)}{=} \left[\frac{4}{5}-\frac{1}{4}\right]R_{yy}(0) = \frac{11}{20}R_{yy}(0)$$

$$\underline{R_{xx}(2)} = \frac{11}{20}\cdot\frac{80}{27} = \underline{\frac{44}{27}}$$

Check:

(1.2) $\frac{80}{27} = \frac{64}{27} - \frac{1}{4}\frac{44}{27} + 1$ ✓

(1.3) $\frac{64}{27} = \frac{80}{27} - \frac{1}{4}\cdot\frac{64}{27}$ ✓

(1.4) $\frac{44}{27} = \frac{64}{27} - \frac{1}{4}\frac{80}{27}$ ✓

7. $X(n) = Z(n) + \phi_1 X(n-1) + \phi_2 X(n-2)$

Let $k \geq 1$:

Multiplying both sides with $X(n-k)$ and taking expectation yields

$$E[X(n)X(n-k)] =$$

$$= E[Z(n)X(n-k)] + \phi_1 E[X(n-1)X(n-k)] + \phi_2 E[X(n-2)X(n-k)]$$

or equivalently

$$\underline{R_{XX}(k) = \phi_1 R_{XX}(k-1) + \phi_2 R_{XX}(k-2)} \qquad k \geq 1 \quad (1.7)$$

Inserting the values of ϕ_1 and ϕ_2 we obtain

$$\underline{R_{XX}(k) = R_{XX}(k-1) - \frac{1}{4} R_{XX}(k-2)} \qquad k \geq 1$$

Notice that the Yule-Walker equations (1.3) and (1.4) coincide with (1.7) for $k = 1$ and $k = 2$ respectively

1.8 $\sigma_X^2 = E[X(n)^2] = R_{XX}(0) = \frac{80}{27}$

1.9

$$R_{XX}(k) = \int_{-1/2}^{+1/2} S_{XX}(f) \exp(j 2\pi k f)\, df$$

$k=0$:

$$\sigma_X^2 = R_{XX}(0) = \int_{-1/2}^{+1/2} S_{XX}(f)\, df$$

Hence

$$\int_{-1/2}^{+1/2} S_{XX}(f)\, df = \frac{80}{27}$$

1.2 Let $Z(n) = \delta(n) = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$ unit impulse at $n=0$.

Then $Y(n) = h(n)$ is the impulse response of the recursive filter. $h(n)$ satisfies

$$h(n) = 0 \qquad n < 0$$

$$h(n) = \delta(n) + \phi_1 h(n-1) + \phi_2 h(n-2)$$

Inserting the values $\phi_1 = 1$ and $\phi_2 = -\frac{1}{4}$ and computing for $n = 0, 1, \dots, 5$ yields

$h(0) = 1 + 0 - \frac{1}{4} \cdot 0 = 1$

$h(1) = 0 + 1 - \frac{1}{4} \cdot 0 = 1$

$h(2) = 0 + 1 - \frac{1}{4} \cdot 1 = \frac{3}{4}$

$h(3) = 0 + \frac{3}{4} - \frac{1}{4} \cdot 1 = \frac{1}{2}$

$h(4) = 0 + \frac{1}{2} - \frac{1}{4} \cdot \frac{3}{4} = \frac{5}{16}$

$h(5) = 0 + \frac{5}{16} - \frac{1}{4} \cdot \frac{1}{2} = \frac{3}{16}$

Problem 2 BPAM Transmission

2.1 BPAM detection as a binary hypothesis testing problem

$$H_0 : \quad u = '1'$$
$$H_1 : \quad u = '0'$$

Received signal under both hypotheses;

$$H_0 : \quad Y = +A + N$$
$$H_1 : \quad Y = -A + N$$

2.2 $$f(y|H_0) = \frac{1}{\sqrt{2\pi}\,\sigma_N} \exp\left\{-\frac{1}{2\sigma_N^2}(y-A)^2\right\}$$

$$f(y|H_1) = \frac{1}{\sqrt{2\pi}\,\sigma_N} \exp\left\{-\frac{1}{2\sigma_N^2}(y+A)^2\right\}$$

2.3 $$l(y) = \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right)$$

$$= -\frac{1}{2\sigma_N^2}(y+A)^2 + \frac{1}{2\sigma_N^2}(y-A)^2$$

$$= \frac{1}{2\sigma_N^2}\left[(y-A)^2 - (y+A)^2\right]$$

$$= \frac{1}{2\sigma_N^2}\left[2y \cdot (-2A)\right]$$

$$\underline{l(y) = -\frac{2A}{\sigma_N^2} \cdot y}$$

2.4 MAP rule:

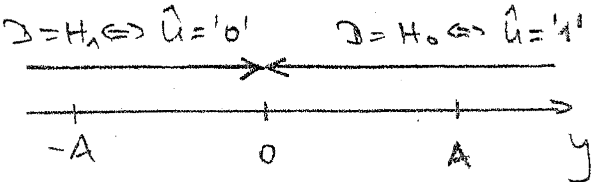
$$l(y) = -\frac{2A}{\sigma_N^2}\, y \overset{H_1}{\underset{H_0}{\gtrless}} \ln\left(\frac{P[H_0]}{P[H_1]}\right).$$

or equivalently

$$y \overset{H_0}{\underset{H_1}{\gtrless}} \frac{\sigma_N^2}{2A} \ln \frac{P[H_1]}{P[H_0]} \overset{\eta = \frac{A^2}{\sigma_N^2}}{=} \frac{A}{2\eta} \ln \frac{P[H_1]}{P[H_0]}$$

a) $P[H_0] = P[H_1] = 1/2$:

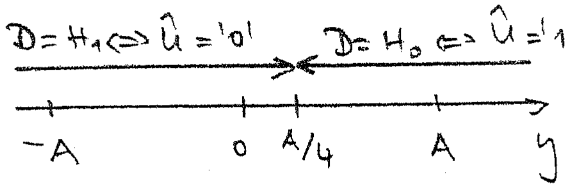
$$y \overset{H_0}{\underset{H_1}{\gtrless}} 0$$



b) $P[H_1] = e^5 P[H_0]$

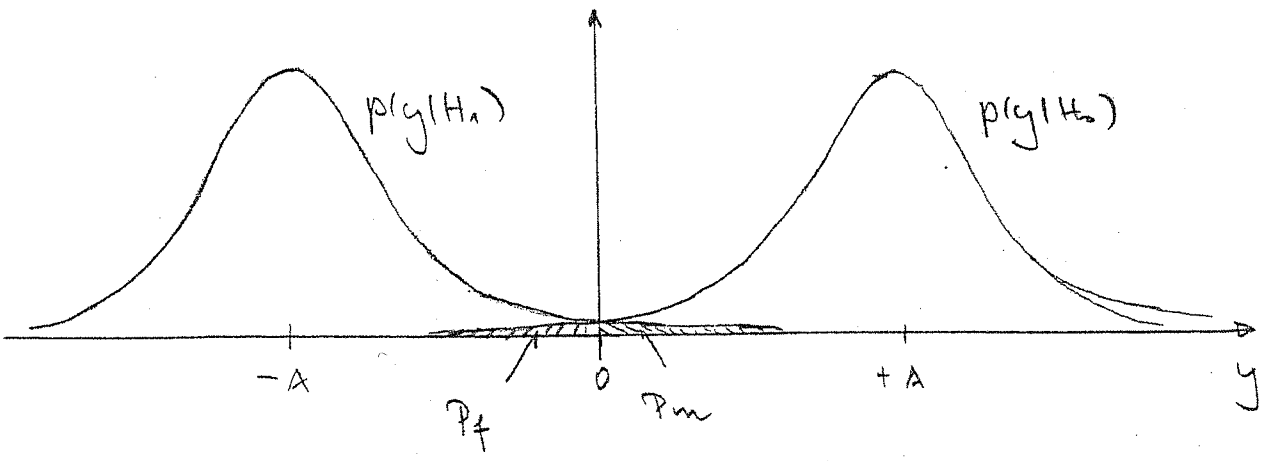
$$y \overset{H_0}{\underset{H_1}{\gtrless}} \frac{A}{2\eta} \ln(e^5) \overset{\eta = 10}{\downarrow \atop =} \frac{A}{20} 5$$

$$y \overset{H_0}{\underset{H_1}{\gtrless}} \frac{A}{4}$$

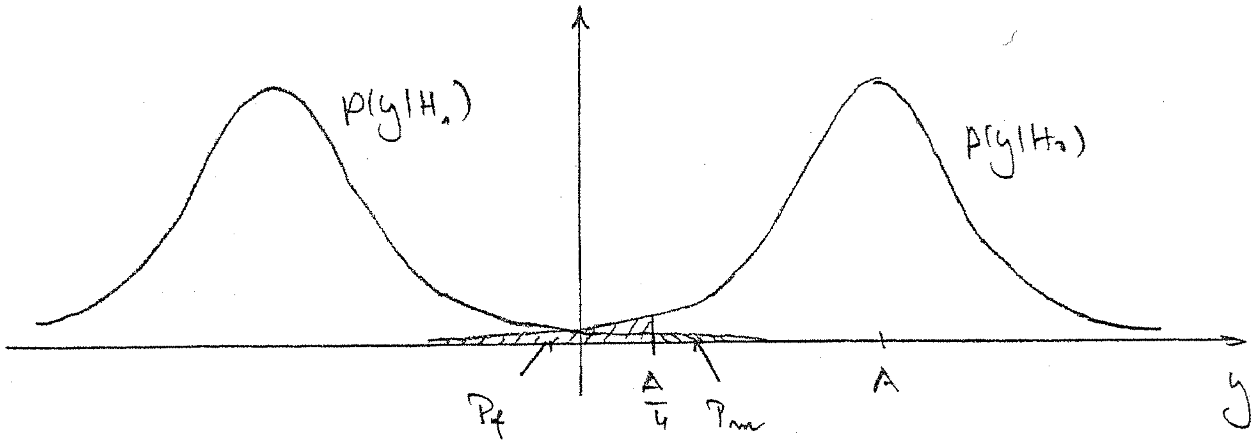


2.5.

a)



b)



2.6

a) $P_f = P_m = \int_0^{\infty} p(y|H_1)\, dy$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_N} \int_0^{\infty} \exp\left(-\frac{1}{2\sigma_N^2}(y+A)^2\right) dy$$

$u = \frac{y+A}{\sigma_N}$

$$= \frac{1}{\sqrt{2\pi}} \int_{A/\sigma_N}^{\infty} \exp\left(-\frac{1}{2} u^2\right) du$$

$$= Q\left(\frac{A}{\sigma_N}\right)$$

$\underline{P_f = P_m = Q(\sqrt{\gamma})}$ $= Q(\sqrt{10}) = 7.83 \cdot 10^{-4}$

b) $P_f = \int_{-\infty}^{\frac{A}{4}} p(y|H_0)\,dy$

$= \frac{1}{\sqrt{2\pi}\sigma_n} \int_{-\infty}^{\frac{A}{4}} \exp\left(-\frac{1}{2\sigma_n^2}(y-A)^2\right) dy$

$\overset{v=\frac{y-A}{\sigma_n}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\frac{3}{4}\frac{A}{\sigma_n}} \exp\left(-\frac{1}{2}v^2\right) dv$

$\underline{P_f = Q\left(\frac{3}{4}\sqrt{\eta}\right)}$ $= Q\left(\frac{3}{4}\sqrt{10}\right) = 8.85 \cdot 10^{-3}$

$P_m = \int_{A/4}^{\infty} p(y|H_1)\,dy$

$\overset{u=\frac{y+A}{\sigma_n}}{=} \frac{1}{\sqrt{2\pi}} \int_{\frac{5}{4}\frac{A}{\sigma_n}}^{\infty} \exp\left(-\frac{u^2}{2}\right) du$

$\underline{P_m = Q\left(\frac{5}{4}\sqrt{\eta}\right)}$ $= Q\left(\frac{5}{4}\sqrt{10}\right) = 3.86 \cdot 10^{-5}$

2.7. $P_e = P_f \cdot P[H_0] + P_m P[H_1]$

a) $\underline{P_e = Q(\sqrt{\eta})}$ $= 7.83 \cdot 10^{-3}$

b) $P[H_1] = e^5 P[H_0]$, $P[H_0] + P[H_1] = 1$

$P[H_0] = (1+e^5)^{-1}$ $P[H_1] = 1-(1+e^5)^{-1}$

$\underline{P_e = Q\left(\frac{3}{4}\sqrt{10}\right)(1+e^5)^{-1} + Q\left(\frac{5}{4}\sqrt{10}\right)\left[1-(1+e^5)^{-1}\right]}$ (2.1)

$= 9.76 \cdot 10^{-5}$

2.8 In this case, $P_m = P_f = Q(\sqrt{10})$

$\Rightarrow \tilde{P}_e = P_m = Q(\sqrt{10}) > P_e$ in (2.1)