Stochastic Processes II (FP-7.5):

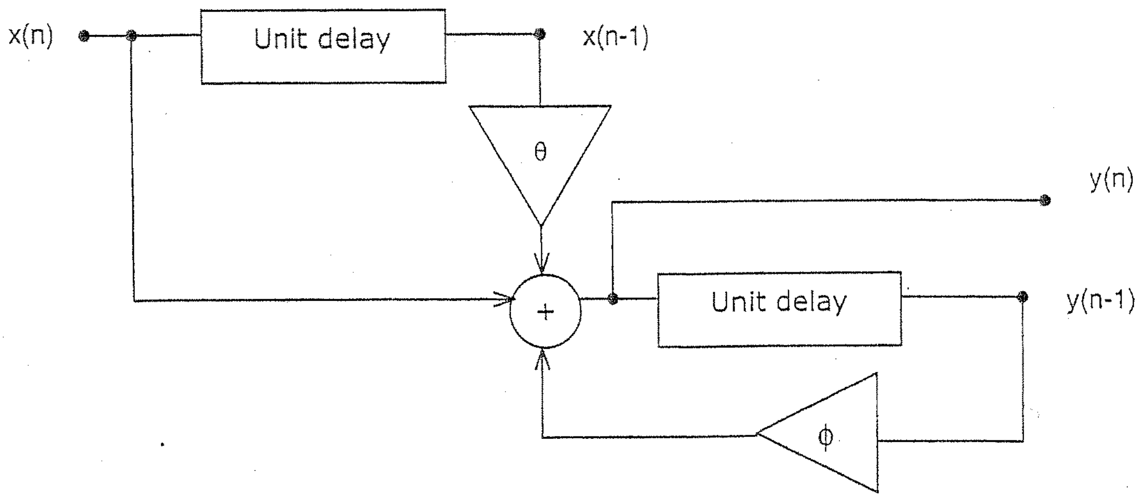
Written Examination

Date and Time: Monday, Jan. 14, 2002, 09.00-11.00.

Hints: Try to solve as many items in the problems as you can. All items have the same weight. Thus, should you encounter some difficulty in solving one question, then skip it and go to the next.

Problem 1: Linear system and ARMA(1,1) process

Let us consider the following linear system



the input-output relationship of which is given by

$$y(n) = \phi y(n-1) + x(n) + \theta x(n-1). \tag{1}$$

1.1. Derive and plot the impulse response $h(n)$ of the linear system for the initialization $x(n) = y(n) = 0$, $n < 0$.

1.2. Determine the range of values of ϕ and θ for which the linear system is stable.

From now on, we assume that $\phi = \theta = \frac{1}{2}$.

1.3. Derive the transfer function $H(f)$ of the linear system.

1.4. Compute and plot the amplitude spectrum $|H(f)|$ of the linear system.

Let us now assume that the input sequence is a white noise sequence $\{X(n)\}$ with unit variance, i.e.

$$\mathrm{E}[X(n)] = 0,\ R_{XX}(k) = \mathrm{E}[X(n)X(n+k)] = \sigma_X^2 \delta(k) \text{ with } \sigma_X^2 = 1.$$

1.5. Show that $\{Y(n)\}$ is wide-sense stationary only if $\mathrm{E}[Y(n)] = 0$ for any n.
Hint: Make use of (1).

1.6. Derive and plot the power spectrum $S_{YY}(f)$ of $\{Y(n)\}$.

1.7. What is the interpretation of the surface under the graph of $S_{YY}(f)$, i.e. the quantity $\int_{-1/2}^{+1/2} S_{YY}(f)df$?

Notice: You do not have to calculate this integral, but only to specify the manner it can be interpreted.

1.8. Show that the crosscorrelation of $X(n)$ and $Y(n)$ is given by

$$\mathrm{E}[X(n)Y(n)] = \sigma_X^2 = 1.$$

Hint: Make use of (1) and notice that because $\{X(n)\}$ is white

$$\mathrm{E}[X(n)Y(n-l)] = 0,\ l > 0.$$

The latter result says that at any time n, $X(n)$ and any past value $Y(n-l)$, $l > 0$, of the output sequence $\{Y(n)\}$ are uncorrelated.

1.9. Compute the variance $\sigma_Y^2 = \mathrm{E}\left[Y(n)^2\right]$ of $\{Y(n)\}$.

Hint: Make use of (1) and the results given in Item 1.8.

1.10. Compute $R_{YY}(0)$.

1.11. Compute a recursive expression for $R_{YY}(k)$ for $k \geq 1$.

The expression is recursive in the sense that $R_{YY}(k)$ is expressed as a function of $R_{YY}(k-1)$.

Hint: Replace the term $Y(n)$ in $R_{YY}(k) = \mathrm{E}[Y(n-k)Y(n)]$ by (1) and make use of the results of Item 1.8.

Problem 2: Linear minimum mean squared error estimation and Wiener filter

Let $\{Y(n)\}$ denote a zero-mean WSS sequence with autocorrelation function

$$R_{YY}(k) = \left(\frac{4}{5}\right)^{|k|}.$$

We observe the sequence $\{X(n)\}$, where

$$X(n) = Y(n) + W(n),$$

with $\{W(n)\}$ being a white noise sequence with variance $\sigma_W^2 = 1/5$.

2.1. Find the autocorrelation $R_{XX}(k)$ of $\{X(n)\}$.

2.2. Derive the cross-correlation $R_{XY}(k)$ of $\{X(n)\}$ and $\{Y(n)\}$.

2.3. We consider the two-step linear predictor $\hat{Y}(n+2) = \sum_{m=0}^{2} h(m)X(n-m)$.

Show by applying the orthogonality principle that the above two-step linear predictor is optimum if and only if its coefficients satisfy the linear equation system

$$\begin{bmatrix} 16/25 \\ 64/125 \\ 256/625 \end{bmatrix} = \begin{bmatrix} 6/5 & 4/5 & 16/25 \\ 4/5 & 6/5 & 4/5 \\ 16/25 & 4/5 & 6/5 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}.$$

2.4. Compute the values of $h(m)$, $m = 0,1,2$.

Hint:

$$\begin{bmatrix} 6/5 & 4/5 & 16/25 \\ 4/5 & 6/5 & 4/5 \\ 16/25 & 4/5 & 6/5 \end{bmatrix}^{-1} = \begin{bmatrix} 625/406 & -25/29 & -50/203 \\ -25/29 & 115/58 & -25/29 \\ -50/203 & -25/29 & 625/406 \end{bmatrix} \approx \begin{bmatrix} 1.54 & -0.86 & -0.25 \\ -0.86 & 1.98 & -0.86 \\ -0.25 & -0.86 & 1.54 \end{bmatrix}$$

2.5. Compute the prediction error $\mathrm{E}\left[\left(\hat{Y}(n+2) - Y(n+2)\right)^2\right]$.

2.6. Compute the transfer function of the non-causal discrete Wiener filter for the estimation of $\{Y(n)\}$ based on the observation of $\{X(n)\}$.

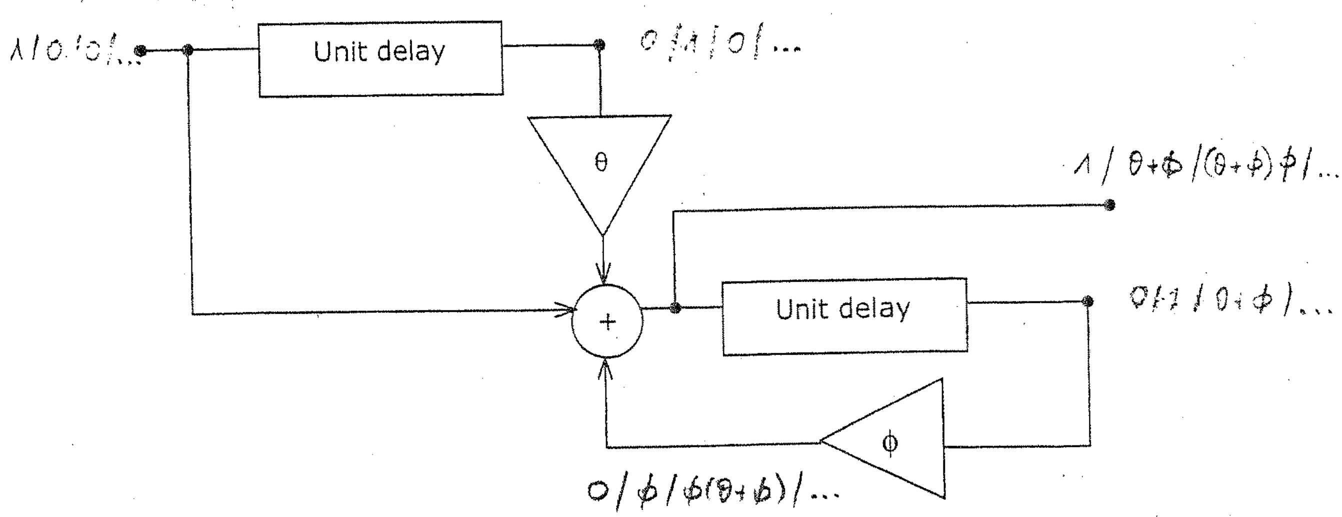
Hint: Compare the autocorrelation function of $\{Y(n)\}$ with that of an AR(1) process.

Solutions of the written examination 13/01/02

FP - 7.5 Stochastic Processes II

Problem 1 Linear system / ARMA (1,1) process

1. Impulse response :



$$h(n) = \begin{cases} 0 & ; \ n < 0 \\ 1 & ; \ n = 0 \\ (\theta+\phi)\,\phi^{n-1} & ; \ n \geq 1 \end{cases}$$

1.2 Stability condition:

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

The above condition is satisfied if, and only if,

$$|\phi| < 1$$

In this case,

$$\sum_{n=0}^{\infty} |h(n)| = 1 + |\theta+\phi| \sum_{n=1}^{\infty} |\phi|^{n-1}$$

$$= 1 + \frac{|\theta+\phi|}{1-|\phi|}$$

The is no restriction on θ for the system to be stable.

In summary,

$$\left.\begin{array}{l}\text{The linear system}\\ \text{is stable}\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}|\phi| < 1\\ \theta \in \mathbb{R}\end{array}\right.$$

1.3 Transfer function:

From (1),

$$Y(f) = \phi \exp(-j2\pi f)\,Y(f) + X(f) + \theta \exp(-j2\pi f)\,X(f)$$

$$Y(f)\left[1 - \phi \exp(-j2\pi f)\right] = \left[1 + \theta \exp(-j2\pi f)\right] X(f)$$

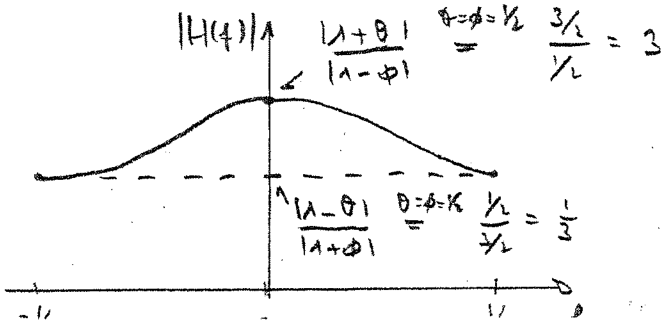
$$H(f) = \frac{Y(f)}{X(f)} = \frac{1+\theta \exp(-j2\pi f)}{1-\phi \exp(-j2\pi f)} \overset{\theta=\phi=1/2}{=} \frac{1+0.5\exp(-j2\pi f)}{1-0.5\exp(-j2\pi f)}$$

1.4 Amplitude spectrum

$$|H(f)| = \frac{|1+\theta \exp(-j2\pi f)|}{|1-\phi \exp(-j2\pi f)|}$$

$$= \frac{\sqrt{(1+\theta\cos(2\pi f))^2 + (\theta \sin(2\pi f))^2}}{\sqrt{(1-\phi\cos(2\pi f))^2 + (\phi \sin(2\pi f))^2}}$$

$$= \sqrt{\frac{1+\theta^2+2\theta\cos(2\pi f)}{1+\phi^2-2\phi\cos(2\pi f)}} \overset{\theta=\phi=1/2}{=} \sqrt{\frac{5/4+\cos(2\pi f)}{5/4-\cos(2\pi f)}}$$



1.5 Wide-sense stationary output

Taking the expectation on both sides of (1) yields

$$E[Y(n)] = \phi E[Y(n-1)] + \underbrace{E[X(n)]}_{=0} + \theta \underbrace{E[X(n-1)]}_{=0}$$

If $Y(n)$ is wide-sense stationary

$$E[Y(n)] = E[Y(n-1)] = \mu_Y$$

Inserting above yields

$$(1-\phi)\,\mu_Y = 0$$

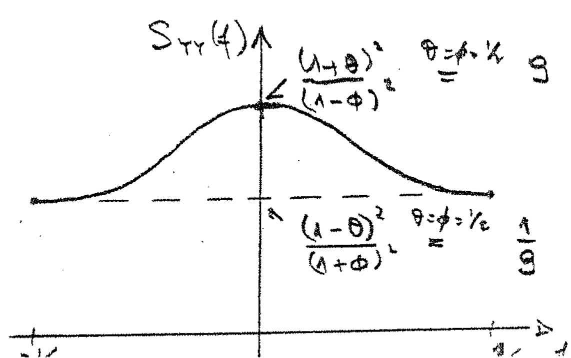
The above condition is satisfied for any $\phi \in (-1,+1)$ if, and only if,

$$\mu_Y = 0$$

1.6. Power spectrum of $\{Y(n)\}$:

$$S_{YY}(f) = |H(f)|^2 \cdot \underbrace{S_{XX}(f)}_{=1}$$

$$= \frac{1+\theta^2+2\theta\cos(2\pi f)}{1+\phi^2-2\phi\cos(2\pi f)} \overset{\theta=\phi=1/2}{=} \frac{\frac{5}{4}+\cos(2\pi f)}{\frac{5}{4}-\cos(2\pi f)}$$



1.7: Surface under $S_{YY}(f)$:

$$\int_{-1/2}^{+1/2} S_{YY}(f)\, df = \sigma_Y^2$$

1.8 Cross correlation of $X(n)$ and $Y(n)$.

$$E[X(n)Y(n)] \overset{(1)}{=} E[X(n)(\phi Y(n-1) + X(n) + \theta X(n-1))]$$

$$= \phi \underbrace{E[X(n)Y(n-1)]}_{=0} + \underbrace{E[X(n)^2]}_{=\sigma_X^2 = 1} + \theta \underbrace{E[X(n)X(n-1)]}_{=0}$$

$$= \sigma_X^2 = 1$$

1.9. Variance of $\{Y(n)\}$:

$$\sigma_Y^2 = E[Y(n)^2] \overset{(1)}{=} E[(\phi Y(n-1) + X(n) + \theta X(n-1))^2]$$

$$= \phi^2 \underbrace{E[Y(n-1)^2]}_{=\sigma_Y^2} + \underbrace{E[X(n)^2]}_{=\sigma_X^2} + \theta^2 \underbrace{E[X(n-1)^2]}_{=\sigma_X^2}$$

$$+ 2\phi \underbrace{E[Y(n-1)X(n)]}_{\substack{\text{Item 1.8}\\ =0}} + 2\phi\theta \underbrace{E[Y(n-1)X(n-1)]}_{\sigma_X^2}$$

$$(1-\phi^2)\sigma_Y^2 = (1 + \theta^2 + 2\phi\theta)\sigma_X^2$$

$$\sigma_Y^2 = \frac{1+\theta^2+2\phi\theta}{1-\phi^2}\sigma_X^2 \overset{\theta=\phi=1/2}{=} \frac{7/4}{3/4} = \frac{7}{3}$$

1.10. $R_{YY}(0) = \sigma_Y^2 = 7/3$

1.11. Recursive expression for $R_{YY}(k)$, $k \geq 1$.

$$R_{YY}(k) = E[Y(n-k)Y(n)]$$

$$\overset{(1)}{=} \phi \underbrace{E[Y(n-k)Y(n-1)]}_{R_{YY}(k-1)} + \underbrace{E[Y(n-k)X(n)]}_{=0} + \theta \underbrace{E[Y(n-k)X(n-1)]}_{= \begin{cases} \sigma_X^2; k= \\ 0; k> \end{cases}}$$

Hence,

(i) $R_{YY}(1) = \phi R_{YY}(0) + \theta \sigma_X^2$
$= \phi \sigma_Y^2 + \theta \sigma_X^2 \overset{\theta=\phi=1/2}{=} \frac{1}{2}\sigma_Y^2 + \frac{1}{2}\sigma_X^2$

(ii) $R_{YY}(k) = \phi R_{YY}(k-1) \quad ; \quad k > 1$

$\overset{\phi=1/2}{=} \frac{1}{2} R_{YY}(k-1) \quad , \quad k > 1$

Problem 2:

2.1 Autocorrelation of $X(n)$

$$R_{XX}(k) = E[X(n)X(n+k)]$$

$$= E[Y(n)Y(n+k)] + E[W(n)W(n+k)]$$

$$= R_{YY}(k) + R_{WW}(k)$$

$$= \left(\frac{4}{5}\right)^{|k|} + \frac{1}{5}\delta(k)$$

2.2 Cross-correlation $R_{XY}(k)$:

$$R_{XY}(k) = E[X(n)Y(n+k)]$$

$$= E[(Y(n) + W(n))\,Y(n+k)]$$

$$= E[Y(n)Y(n+k)] + E[W(n)Y(n+k)]$$

$$= R_{YY}(k)$$

$$= \left(\frac{4}{5}\right)^{|k|}$$

2.3 Coefficients of the two-step predictor:

Orthogonality principle:

$$E[(Y(n+2) - \hat{Y}(n+2))\,X(n-m')] = 0 \qquad m' = 0,1,2$$

$$E[Y(n+2)X(n-m')] = \sum_{m=0}^{2} h(m)\,E[X(n-m)\,X(n-m')]$$

$$R_{XY}(m'+2) = \sum_{m=0}^{2} h(m)\,R_{XX}(m-m') \qquad m'=0,1,2$$

$$\begin{bmatrix} R_{xy}(2) \\ R_{xy}(3) \\ R_{xy}(4) \end{bmatrix} = \begin{bmatrix} R_{xx}(0) & R_{xx}(1) & R_{xx}(2) \\ R_{xx}(1) & R_{xx}(0) & R_{xx}(1) \\ R_{xx}(2) & R_{xx}(1) & R_{xx}(0) \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$$

$$\begin{bmatrix} \left(\frac{4}{5}\right)^2 \\ \left(\frac{4}{5}\right)^3 \\ \left(\frac{4}{5}\right)^4 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} & \frac{4}{5} & \frac{16}{25} \\ \frac{4}{5} & \frac{6}{5} & \frac{4}{5} \\ \frac{16}{25} & \frac{4}{5} & \frac{6}{5} \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$$

2.4 Coefficients of the optimum predictor

$$\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} \frac{625}{406} & -\frac{25}{29} & -\frac{50}{203} \\ -\frac{25}{29} & \frac{145}{58} & -\frac{25}{29} \\ -\frac{50}{203} & -\frac{25}{29} & \frac{625}{406} \end{bmatrix} \begin{bmatrix} \left(\frac{4}{5}\right)^2 \\ \left(\frac{4}{5}\right)^3 \\ \left(\frac{4}{5}\right)^4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4430 \\ 0.1103 \\ 0.0315 \end{bmatrix}$$

2.5 Prediction error:

$$E[(y(n+2) - \hat{y}(n+2))^2] = R_{yy}(0) - \sum_{m=0}^{2} h(m) R_{xy}(m+2)$$

$$= 1 - [0.4430 \quad 0.1103 \quad 0.315] \left[\left(\tfrac{4}{5}\right)^2 \quad \left(\tfrac{4}{5}\right)^3 \quad \left(\tfrac{4}{5}\right)^4 \right]^T$$

$$= 0.6471$$

2.6. Non-causal Wiener filter

Transfer function of the non-causal Wiener filter:

$$H(f) = \frac{S_{XY}(f)}{S_{XX}(f)} = \frac{S_{YY}(f)}{S_{XX}(f)}$$

$$S_{YY}(f) = \sum_{k=-\infty}^{+\infty} R_{YY}(k) \exp(-j2\pi f k)$$

$$= \sum_{k=-\infty}^{+\infty} \theta^{|k|} \exp(-j2\pi f k) \qquad \theta = \frac{4}{5}$$

$$= \sum_{k\leq 0} \theta^{|k|} \exp(-j2\pi f k) + \sum_{k \geq 0} \theta^{|k|} \exp(-j2\pi f k) - 1$$

$$= \sum_{k=0}^{\infty} \theta^{k} \exp(j2\pi f k) + \sum_{k=0}^{\infty} \theta^{k} \exp(-j2\pi f k) - 1$$

$$= \frac{1}{1-\theta\exp(j2\pi f)} + \frac{1}{1-\theta\exp(-j2\pi f)} - 1$$

$$= \frac{1}{|1-\theta\exp(-j2\pi f)|^2} \cdot \Big[1-\theta\exp(j2\pi f) + 1 - \theta\exp(-j2\pi f) - \big(1 - 2\mathrm{Re}\{\theta\exp(-j2\pi f)\} + \theta^2\big)\Big]$$

$$= \frac{1-\theta^2}{|1-\theta\exp(-j2\pi f)|^2}$$

$$S_{XX}(f) = S_{YY}(f) + S_{WW}(f)$$

$$= \frac{1-\theta^2}{|1-\theta\exp(-j2\pi f)|^2} + \sigma_W^2$$

$$H(f) = \frac{\dfrac{1-\theta^2}{|1-\theta\exp(-j2\pi f)|^2}}{\dfrac{1-\theta^2}{|1-\theta\exp(-j2\pi f)|^2} + \sigma_w^2}$$

$$= \frac{1-\theta^2}{1-\theta^2 + \sigma_w^2\,|1-\theta\exp(-j2\pi f)|^2}$$

$$H(f) = \frac{1}{1 + \dfrac{\sigma_w^2}{1-\theta^2}\,|1-\theta\exp(-j2\pi f)|^2}$$

$$\begin{bmatrix} \frac{6}{5} & \frac{4}{5} & \frac{16}{25} \\ \frac{4}{5} & \frac{6}{5} & \frac{4}{5} \\ \frac{16}{25} & \frac{4}{5} & \frac{6}{5} \end{bmatrix}^{-1} = 25 \begin{bmatrix} 30 & 20 & 16 \\ 20 & 30 & 20 \\ 16 & 25 & 30 \end{bmatrix}^{-1}$$

$$= 25 \cdot \begin{bmatrix} \frac{500}{8120} & -\frac{20}{580} & -\frac{80}{8120} \\ -\frac{20}{580} & \frac{46}{580} & -\frac{20}{580} \\ -\frac{80}{8120} & -\frac{20}{580} & \frac{500}{8120} \end{bmatrix} =$$

$$= 25 \begin{bmatrix} \frac{25}{406} & -\frac{1}{29} & -\frac{\cancel{4}\ 2}{\cancel{406}\ 203} \\ -\frac{1}{29} & \frac{23}{290} & -\frac{1}{29} \\ -\frac{2}{203} & -\frac{1}{29} & \frac{25}{406} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{625}{406} & -\frac{25}{29} & -\frac{50}{203} \\ -\frac{25}{29} & \frac{\cancel{575}\ 115}{\cancel{290}\ 58} & -\frac{25}{29} \\ -\frac{50}{203} & -\frac{25}{29} & \frac{625}{406} \end{bmatrix}$$