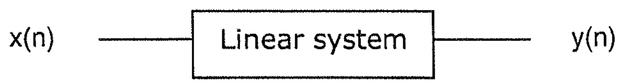
Stochastic Processes II (FP-7.5):

Written Examination

Date and Time: Tuesday, Jan. 14, 2003, 09.00-11.00.

Hints: Try to solve as many items in the problems as you can. All items have the same weight. Thus, should you encounter some difficulty in solving one question, then skip it and go to the next.

Problem 1: Linear systems and WSS processes
Let us consider the non-causal linear system,



the input-output relationship of which is given by

$$y(n)=\frac{1}{\sqrt{2}}x(n-5)+\frac{1}{\sqrt{2}}x(n+5).$$

1.1. Derive and plot the impulse response $h(n)$ of the linear system.
1.2. Derive the transfer function $H(f)$ of the linear system.
1.3. Compute and plot the amplitude spectrum $|H(f)|$ of the linear system.
1.4. Compute the autocorrelation function $R_{hh}(k)=\sum_n h(n)h(n+k)$ of the impulse response $h(n)$.
1.5. Compute the Fourier transform of $R_{hh}(k)$.

Let us now assume that the input sequence is a zero-mean WSS process $\{X(n)\}$ with autocorrelation function,

$$R_{XX}(k)=\mathrm{E}[X(n)X(n+k)]=\begin{cases}1 & ; \quad k=0\\ 1/3 & ; \quad |k|=1\\ 0 & ; \quad |k|>1\end{cases}$$

1.6. Compute the variance of $\{X(n)\}$.
1.7. Compute the power spectrum $S_{XX}(f)$ of $\{X(n)\}$.
1.8. Derive the autocorrelation function $R_{YY}(k)=\mathrm{E}[Y(n)Y(n+k)]$ of the output process $\{Y(n)\}$.
1.9. Derive and plot the power spectrum $S_{YY}(f)$ of $\{Y(n)\}$.
1.10. Compute the variance of $\{Y(n)\}$.

Problem 2: Linear minimum mean squared error estimation and Wiener filter
Let $\{Y(n)\}$ denote a zero-mean WSS sequence with autocorrelation function

$$R_{YY}(k) = \left(\frac{4}{5}\right)^{|k|}.$$

We observe the sequence $\{X(n)\}$, where

$$X(n) = A(n)Y(n) + W(n),$$

where

- $\{W(n)\}$ is a white noise sequence with variance $\sigma_W^2 = 1/10$.
- $\{A(n)\}$ denotes a WSS sequence with expectation $\mathrm{E}[A(n)] = 1$ and autocorrelation function $R_{AA}(k) = \left(\frac{9}{10}\right)^{|k|}$.

  $\{A(n)\}$ represents the attenuation occurring in the channel.
- The three processes $\{A(n)\}$, $\{Y(n)\}$, and $\{W(n)\}$ are independent. As a consequence, if $\{U(n)\}$ and $\{V(n)\}$ represent any two of these processes, then the autocorrelation function of the "product" process $\{U(n)V(n)\}$ is equal to $R_{UU}(k)R_{VV}(k)$.

2.1. Find the autocorrelation $R_{XX}(k)$ of $\{X(n)\}$.

2.2. Derive the cross-correlation $R_{XY}(k)$ of $\{X(n)\}$ and $\{Y(n)\}$.

2.3. We consider the linear estimator $\hat{Y}(n) = \sum_{m=0}^{2} h(m)X(n-m)$.

Show by applying the orthogonality principle that the above two-step linear predictor is optimum if and only if its coefficients satisfy the linear equation system

$$\begin{bmatrix} 1 \\ 4/5 \\ 16/25 \end{bmatrix} = \begin{bmatrix} 11/10 & 18/25 & 324/625 \\ 18/25 & 11/10 & 18/25 \\ 324/625 & 18/25 & 11/10 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}.$$

2.4. Compute the values of $h(m)$, $m = 0,1,2$.

$$\textit{Hint:} \begin{bmatrix} 11/10 & 18/25 & 324/625 \\ 18/25 & 11/10 & 18/25 \\ 324/625 & 18/25 & 11/10 \end{bmatrix}^{-1} \approx \begin{bmatrix} 1.60 & -0.97 & -0.12 \\ -0.97 & 2.18 & -0.97 \\ -0.12 & -0.97 & 1.60 \end{bmatrix}$$

2.5. Compute the estimation error $\mathrm{E}\left[\left(\hat{Y}(n) - Y(n)^2\right)\right]$.

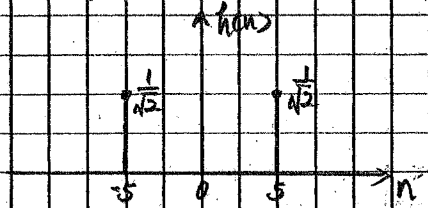
2.6. Compute the transfer function of the non-causal discrete Wiener filter for the estimation of $\{Y(n)\}$ based on the observation of $\{X(n)\}$.

Hint: Compare the autocorrelation functions of $\{Y(n)\}$ and $\{A(n)\}$ with that of an AR(1) process.

Problem 1.

$$y(n) = \frac{1}{\sqrt{2}} x(n-5) + \frac{1}{\sqrt{2}} x(n+5)$$

1.1 $$h(n) = \frac{1}{\sqrt{2}} \delta(n-5) + \frac{1}{\sqrt{2}} \delta(n+5)$$

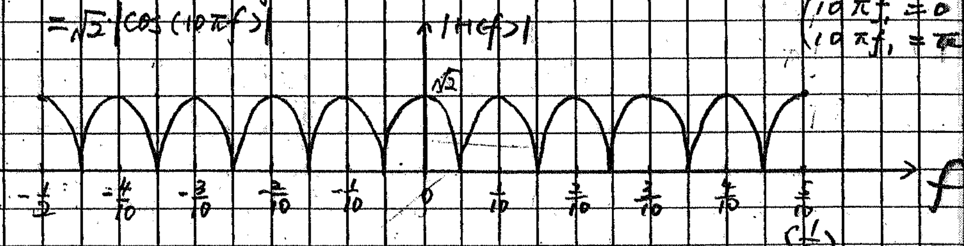


1.2 $$\begin{aligned} H(f) &= \mathcal{F}\{h(n)\} \\ &= \mathcal{F}\left\{ \frac{1}{\sqrt{2}} (\delta(n-5) + \delta(n+5)) \right\} \\ &= \frac{1}{\sqrt{2}} \mathcal{F}\{ \delta(n-5) + \delta(n+5) \} \\ &= \frac{1}{\sqrt{2}} \left[\sum_n \delta(n-5) \cdot \exp(-j2\pi f n) + \sum_n \delta(n+5) \cdot \exp(-j2\pi f n) \right] \\ &= \frac{1}{\sqrt{2}} \left[\exp(-j2\pi f \cdot 5) + \exp(-j2\pi f \cdot (-5)) \right] \\ &= \frac{1}{\sqrt{2}} \left[\cos(10\pi f) - i \cdot \sin(10\pi f) + \cos(10\pi f) + i \cdot \sin(10\pi f) \right] \\ &= \frac{1}{\sqrt{2}} \cdot 2 \cos(10\pi f) \\ &= \sqrt{2} \cdot \cos(10\pi f), \quad |f| < \frac{1}{2} \end{aligned}$$

1.3 $$\begin{aligned} |H(f)| &= |\sqrt{2} \cos(10\pi f)| \\ &= \sqrt{2} |\cos(10\pi f)| \end{aligned}$$

Notice: Period of $|\cos(10\pi f)|$ is $\frac{1}{10}$.

$$\begin{cases} 10\pi f_1 = 0 \Rightarrow f_1 = 0 \\ 10\pi f_1 = \pi \Rightarrow f_1 = \frac{1}{10} \end{cases}$$



1.4 $$\begin{aligned} R_{hh}(k) &= \sum_n h(n) h(n+k) \\ &= \sum_n \left(\frac{1}{\sqrt{2}} \delta(n-5) + \frac{1}{\sqrt{2}} \delta(n+5) \right) \left(\frac{1}{\sqrt{2}} \delta(n+k-5) + \frac{1}{\sqrt{2}} \delta(n+k+5) \right) \\ &= \frac{1}{2} \sum_n \big(\delta(n-5) \delta(n+k-5) + \delta(n+5) \delta(n+k-5) + \delta(n-5) \delta(n+k+5) \\ &\qquad + \delta(n+5) \delta(n+k+5) \big) \\ &= \frac{1}{2} \left[\delta(k) + \delta(k-10) + \delta(k+10) + \delta(k) \right] \\ &= \frac{1}{2} \delta(k-10) + \delta(k) + \frac{1}{2} \delta(k+10) \end{aligned}$$

1.5 $$\begin{aligned} S_{hh}(f) &= \mathcal{F}\{ R_{hh}(k) \} \\ &= \frac{1}{2} \cdot \mathcal{F}\{ \delta(k-10) + 2\delta(k) + \delta(k+10) \} \\ &= \frac{1}{2} \cdot \left[\exp(-j20\pi f) + 2 + \exp(j20\pi f) \right] \\ &= \frac{1}{2} \cdot \left[2 + 2 \cdot \cos(20\pi f) \right] \\ &= 1 + \cos(20\pi f) \end{aligned}$$

1.6 $\sigma_x^2 = R_{xx}(0) = 1$

1.7 $R_{xx}(k) = \frac{1}{3}\delta(k+1) + \delta(k) + \frac{1}{3}\delta(k-1)$

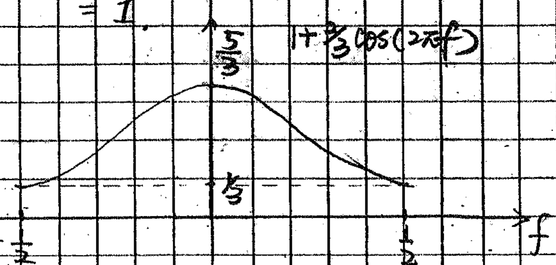
$$S_{xx}(f) = \mathcal{F}\{R_{xx}(k)\}$$
$$= \frac{1}{3}\cdot \exp(j\cdot 2\pi f) + 1 + \frac{1}{3}\exp(-j2\pi f)$$
$$= 1 + \frac{1}{3}\cdot 2\cos(2\pi f), \quad |f| < \frac{1}{2}$$

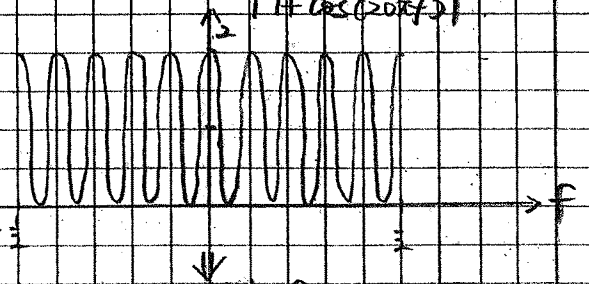
1.8.
$$R_{YY}(k) = R_{hh}(k) * R_{xx}(k)$$
$$= \left[\frac{1}{2}\delta(k-10) + \delta(k) + \frac{1}{2}\delta(k+10)\right] * R_{xx}(k)$$
$$= R_{xx}(k) + \frac{1}{2}R_{xx}(k-10) + \frac{1}{2}R_{xx}(k+10).$$

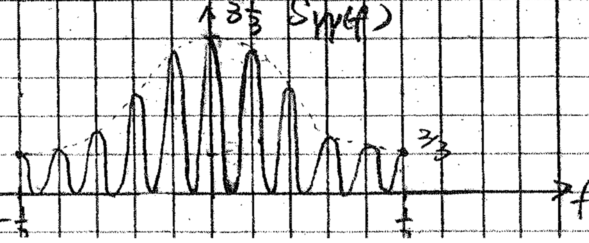
or.
$$R_{YY}(k) = E[Y(n)Y(n+k)]$$
$$= E\left[\frac{1}{\sqrt{2}}(x(n-5) + x(n+5))\cdot\frac{1}{\sqrt{2}}\cdot(x(n+k-5) + x(n+k+5))\right]$$
$$= E\left[\frac{1}{2}(x(n-5) + x(n+5))\cdot(x(n+k-5) + x(n+k+5))\right]$$
$$= \frac{1}{2}\left[R_{xx}(k) + R_{xx}(k+10) + R_{xx}(k-10) + R_{xx}(k)\right]$$
$$= R_{xx}(k) + \frac{1}{2}R_{xx}(k+10) + \frac{1}{2}R_{xx}(k-10).$$

1.9
$$S_{YY}(f) = |H(f)|^2\cdot S_{xx}(f)$$
$$= 2|\cos(10\pi f)|^2(1 + \frac{2}{3}\cos(2\pi f))$$
$$= 2\cdot\frac{1}{2}|1 + \cos 20\pi f|\cdot(1 + \frac{2}{3}\cos(2\pi f))$$
$$= |1 + \cos(20\pi f)|\cdot(1 + \frac{2}{3}\cos(2\pi f))$$

1.10
$$Var(Y(n)) = R_{YY}(k)|_{k=0}$$
$$= R_{xx}(0) + \frac{1}{2}R_{xx}(10) + \frac{1}{2}R_{xx}(-10)$$
$$= 1 + 0 + 0$$
$$= 1.$$







Problem 2.

2.1 $R_{xx}(k) = E[x(n)\,x(n+k)]$

$$= E\{[A(n)Y(n) + W(n)]\cdot[A(n+k)Y(n+k) + W(n+k)]\}$$

$$= E\{A(n)A(n+k)Y(n)Y(n+k) + A(n)Y(n)W(n+k) + W(n)A(n+k)Y(n+k) + W(n)W(n+k)\}$$

$$= R_{AA}(k)\cdot R_{YY}(k) + E[A(n)]\cdot E[Y(n)]\cdot E[W(n+k)] + E[W(n)]\cdot E[A(n+k)]\cdot E[Y(n+k)] + R_{WW}(k)$$

$$= \left(\frac{9}{10}\right)^{|k|}\cdot\left(\frac{4}{5}\right)^{|k|} + 0 + 0 + \sigma_w^2\cdot\delta(k)$$

$$= \left(\frac{18}{25}\right)^{|k|} + \frac{1}{10}\,\delta(k).$$

2.2 $R_{XY}(k) = E[X(n)\cdot Y(n+k)]$

$$= E\{[A(n)Y(n) + W(n)]\cdot Y(n+k)\}$$

$$= E\{A(n)Y(n)Y(n+k) + W(n)\cdot Y(n+k)\}$$

$$= E[A(n)]\,R_{YY}(k) + E[W(n)]\,E[Y(n+k)]$$

$$= 1\cdot R_{YY}(k) + 0$$

$$= \left(\frac{4}{5}\right)^{|k|}.$$

2.3 • orthogonality principle

$$E[(Y(n) - \hat{Y}(n))X(n-k)] = 0,\quad \forall k = 0,1,2$$

$$\Rightarrow\quad E\left\{\left[Y(n) - \sum_{m=0}^{2} h(m)X(n-m)\right]\cdot X(n-k)\right\} = 0$$

$$E\left[Y(n)X(n-k) - \sum_{m=0}^{2} h(m)X(n-m)\cdot X(n-k)\right] = 0$$

$$R_{XY}(k) - \sum_{m=0}^{2} h(m)\,E[X(n-m)X(n-k)] = 0$$

$$R_{XY}(k) - \sum_{m=0}^{2} h(m)\,R_{XX}(m-k) = 0$$

$$R_{XY}(k) = \sum_{m=0}^{2} h(m)\,R_{XX}(m-k)$$

$$\begin{bmatrix} R_{XY}(0) \\ R_{XY}(1) \\ R_{XY}(2) \end{bmatrix} = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & R_{XX}(2) \\ R_{XX}(-1) & R_{XX}(0) & R_{XX}(1) \\ R_{XX}(-2) & R_{XX}(-1) & R_{XX}(0) \end{bmatrix}\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$$

i.e. $$\begin{bmatrix} 1 \\ \frac{4}{5} \\ \frac{16}{25} \end{bmatrix} = \begin{bmatrix} \frac{11}{10} & \frac{18}{25} & \frac{324}{625} \\ \frac{18}{25} & \frac{11}{10} & \frac{18}{25} \\ \frac{324}{625} & \frac{18}{25} & \frac{11}{10} \end{bmatrix}\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$$

2.4

$$\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} \frac{11}{10} & \frac{18}{25} & \frac{324}{625} \\ \frac{18}{25} & \frac{11}{10} & \frac{18}{25} \\ \frac{324}{625} & \frac{18}{25} & \frac{11}{10} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ \frac{4}{5} \\ \frac{16}{25} \end{bmatrix}$$

$$\overset{*}{\approx} \begin{bmatrix} 1.60 & -0.97 & -0.12 \\ -0.97 & 2.18 & -0.97 \\ -0.12 & -0.97 & 1.60 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \frac{4}{5} \\ \frac{16}{25} \end{bmatrix}$$

$$= \begin{bmatrix} 0.7472 \\ 0.1532 \\ 0.1280 \end{bmatrix}$$

Notice:
If the inverse m[...] is accurate, the result of $h(m)$, $m=0$[...] reads:
$$\begin{bmatrix} 0.7480 \\ 0.1532 \\ 0.1290 \end{bmatrix}.$$

2.5

$$E\left[(\hat{Y}(n) - Y(n))^2\right] = R_{YY}(0) - \sum_{m=0}^{2} h(m) R_{XY}(m)$$
$$= 1 - 0.7472 \cdot 1 - 0.1532 \cdot \frac{4}{5} - 0.1280 \cdot \frac{16}{25}$$
$$= 0.0483$$

Notice:
If the $h(m)$ has the value accurate
$E[(\hat{Y}(n) - Y(n))^2]$ reads 0.0589.

2.6.

$$H(f) = \frac{S_{XY}(f)}{S_{XX}(f)}$$

$$S_{XY}(f) = \mathcal{F}\{R_{XY}(k)\}$$
$$= \mathcal{F}\{R_{YY}(k)\}^{**}$$
$$= \sum_k \left(\frac{4}{5}\right)^{|k|} \exp(-j2\pi k f) = \frac{1 - \left(\frac{4}{5}\right)^2}{|1 - \frac{4}{5}\exp(-j2\pi f)|^2}$$

$$S_{XX}(f) = \mathcal{F}\{R_{XX}(k)\}$$
$$= \sum_k \left[\left(\frac{18}{25}\right)^{|k|} + \frac{1}{10}\delta(k)\right] \cdot \exp(-j2\pi k f)$$
$$= \sum_k \left(\frac{18}{25}\right)^{|k|} \exp(-j2\pi k f) + \frac{1}{10} \cdot 1$$
$$= \frac{1 - \left(\frac{18}{25}\right)^2}{|1 - \frac{18}{25}\exp(-j2\pi f)|^2} + \frac{1}{10}$$

$$H(f) = \frac{S_{XY}(f)}{S_{XX}(f)} = \frac{\dfrac{9}{25 \cdot |1 - \frac{4}{5}\exp(-j2\pi f)|^2}}{\dfrac{301}{625 \cdot |1 - \frac{18}{25}\exp(-j2\pi f)|^2} + \frac{1}{10}}$$

** According to the lecture note 1-11,

$$\mathcal{F}\{\phi^{|k|}\} = \frac{1 - \phi^2}{|1 - \phi \cdot \exp(-j2\pi f)|^2}.$$