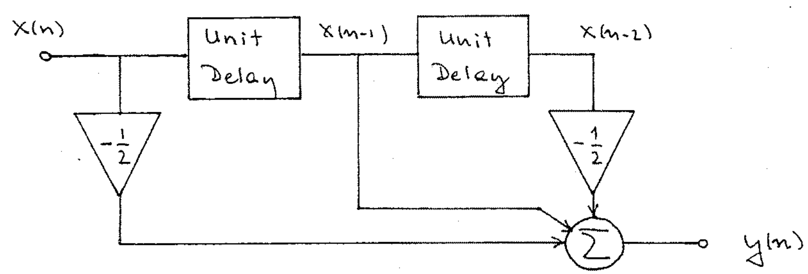
Stochastic Processes II (FP-7.5):

Written Examination

Date and Time: Thursday, Jan. 13, 2000, 09.00-11.00.

Hints: Try to solve as many items in the problems as you can. All items have the same weight. Thus, should you encounter some difficulty in deriving a solution, then skip the question and postpone it to the end after you have addressed all the items that you can easily solve.

Problem 1: Linear systems/MA processes
Let us consider the transversal filter (TF) depicted below:



1.1. Write the equation that relates the output sequence $\{y(n)\}$ to the input sequence $\{x(n)\}$ (input-output relationship of the TF).

1.2. Derive and plot the impulse response $h(n)$ of the TF.

1.3. Derive the transfer function $H(f)$ of the TF.
Plot $|H(f)|$.
Hint: Write $H(f)$ in the form $H(f)=\exp(-j2\pi f)G(f)$.

1.4. Let us now assume that the input sequence is a white noise sequence $\{X(n)\}$, i.e.
$$\mathrm{E}[X(n)]=0 \text{ and } R_{XX}(k)=\mathrm{E}[X(n)X(n+k)]=\delta(k).$$
Compute the expectation $\mathrm{E}[Y(n)]$ of the output $\{Y(n)\}$ of the TF.

1.5. Compute the autocorrelation function $R_{YY}(k)=\mathrm{E}[Y(n)Y(n+k)]$ of $\{Y(n)\}$ and plot it.

1.6. Compute the variance $\sigma_Y^2=\mathrm{E}\left[Y(n)^2\right]$ of $\{Y(n)\}$.

1.7. Derive and plot the power spectrum of $\{Y(n)\}$.

1.8. Compute the variance σ_Y^2 of $\{Y(n)\}$ from its power spectrum.
Hint: $\sin^4(\alpha)=\frac{1}{8}[\cos(4\alpha)-4\cos(2\alpha)+3]$

Problem 2: Linear minimum mean squared error estimation and Wiener filtering

Let $\{Y(n)\}$ denote an AR(1) process with autocorrelation function defined as
$$R_{YY}(k)=\frac{4}{3}\left(\frac{1}{2}\right)^{|k|}.$$
We observe the sequence $\{X(n)\}$, where
$$X(n)=Y(n)+N(n)$$
with $\{N(n)\}$ being a white noise sequence with variance $\sigma_N^2=\mathrm{E}\left[N(n)^2\right]=\frac{1}{3}$.
Moreover, $\{Y(n)\}$ and $\{N(n)\}$ are uncorrelated.

2.1. The AR(1) process $\{Y(n)\}$ satisfies the relation
$$Y(n)=\phi Y(n-1)+Z(n),$$
where $\{Z(n)\}$ is a white process with variance $\sigma_Z^2=\mathrm{E}\left[Z(n)^2\right]$.
Derive the feedback coefficient ϕ and the variance σ_Z^2 from $R_{YY}(k)$.

2.2. Find the autocorrelation $R_{XX}(k)$ of $\{X(n)\}$.

2.3. Derive the cross-correlation $R_{XY}(k)$ of $\{X(n)\}$ and $\{Y(n)\}$.

2.4. Show by applying the orthogonality principle that the coefficients of the linear minimum mean squared estimator of $Y(n)$ based on the observation of $X(n)$, $X(n-1)$, $X(n-2)$
$$\hat{Y}(n)=\sum_{m=0}^{2}h(m)X(n-m)$$
satisfy the linear equation system
$$\begin{bmatrix} 5/3 & 2/3 & 1/3 \\ 2/3 & 5/3 & 2/3 \\ 1/3 & 2/3 & 5/3 \end{bmatrix}\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}=\begin{bmatrix} 4/3 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

2.5. Compute the values of $h(m)$, $m=0,1,2$.
Hint:
$$\begin{bmatrix} 5/3 & 2/3 & 1/3 \\ 2/3 & 5/3 & 2/3 \\ 1/3 & 2/3 & 5/3 \end{bmatrix}^{-1}=\frac{3}{88}\begin{bmatrix} 21 & -8 & -1 \\ -8 & 24 & -8 \\ -1 & -8 & 21 \end{bmatrix}$$

2.6. Compute the residual prediction error $\mathrm{E}\left[\left(Y(n)-\hat{Y}(n)\right)^2\right]$.

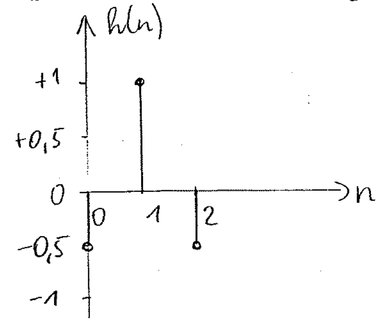
2.7. Compute and plot the transfer function of the non-causal discrete Wiener filter for the estimation of $\{Y(n)\}$ based on the observation of $\{X(n)\}$.

Solutions of the written examination Stochastic Processes II

00/01/11

1. $y(n) = -\frac{1}{2}x(n) + x(n-1) - \frac{1}{2}x(n-2)$

2. $h(n) = -\frac{1}{2}\delta(n) + \delta(n-1) - \frac{1}{2}\delta(n-2)$



3. $\tilde{h}(n) = -\frac{1}{2}\delta(n+1) + \delta(n) - \frac{1}{2}\delta(n-1)$

$\tilde{Y}(f) = \mathcal{F}\{x(n) * \tilde{h}(n)\} = -\frac{1}{2}X(f)\exp(+j2\pi f) +$

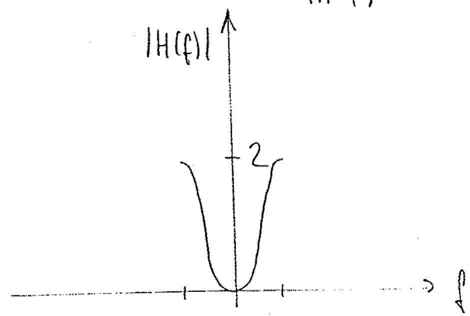
$+ X(f) - \frac{1}{2}X(f)\exp[-j2\pi f] =$

$= X(f) - \frac{1}{2}X(f)[\exp(j2\pi f) + \exp(-j2\pi f)] =$

$= X(f) - X(f)\cdot\cos(2\pi f) = X(f)\cdot[1-\cos(2\pi f)]$

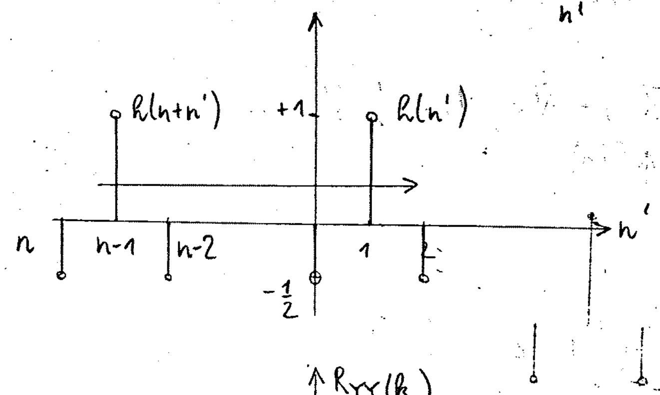
$h(n) = \tilde{h}(n-1) \Rightarrow Y(f) = \exp(-j2\pi f)\cdot\tilde{Y}(f)$

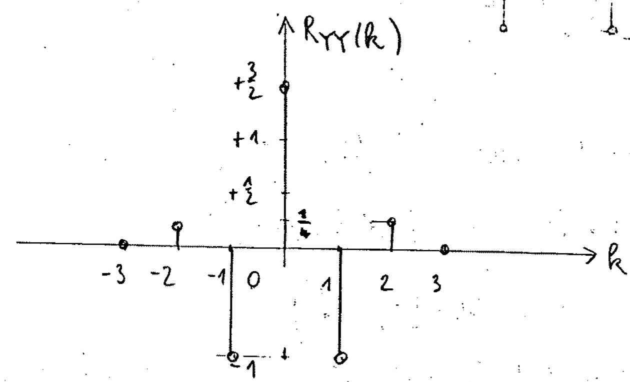
$Y(f) = H(f)\cdot X(f) = \underbrace{\exp(-j2\pi f)}_{\varphi_H(f)}\underbrace{[1-\cos(2\pi f)]}_{|H(f)|}\cdot X(f)$



1.4. $E[Y(n)] = E[-\frac{1}{2}X(n) + X(n-1) - \frac{1}{2}X(n-2)] = \underline{\underline{\emptyset}}$

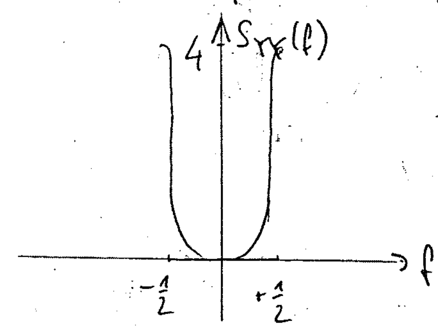
1.5. $R_{YY}(k) = E[Y(n)Y(n+k)] = \sum_{n'} h(n')h(n+n')$





1.6. $\sigma_Y^2 = R_{YY}(0) = \frac{3}{2}$

1.7. $S_{YY}(f) = |H(f)|^2\cdot S_{XX}(f) = [1-\cos(2\pi f)]^2\sigma_X^2 = 4\sin^4(\pi f)$



1.8. $\sigma_Y^2 = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{YY}(f)\,df = \int_{-\frac{1}{2}}^{+\frac{1}{2}}\left[\frac{1}{2}\cos(4\pi f) - 2\cos(2\pi f) + \frac{3}{2}\right]df = \frac{3}{2}$

2.1. $Y(n) = \phi_1 \cdot Y(n-1) + Z(n)$

Yule-Walker equation: $R_{YY}(k) = \phi_1 \cdot R_{YY}(k-1)$

$\phi_1 = \frac{R_{YY}(1)}{R_{YY}(0)} = \frac{4 \cdot 3}{6 \cdot 4} = \underline{\underline{\frac{1}{2}}}$

$Var(Y(n)) = E[Y(n)^2] = \frac{1}{4} \underbrace{E[Y(n-1)^2]}_{E[Y(n)^2]} + E[\underbrace{\cancel{Y(n-1)Z(n)}}_{0}] + E[Z(n)^2]$

$\Rightarrow \underline{\underline{\sigma_z^2}} = E[Z(n)^2] = \frac{3}{4} E[Y(n)^2] = \frac{3}{4} \cdot \frac{4}{3} = \underline{\underline{1}}$

2.2. $R_{XX}(k) = E[X(n)X(n+k)] = E[(Y(n)+N(n)) \cdot$

$\cdot (Y(n+k) + N(n+k))] = E[Y(n)Y(n+k)] + E[N(n) \cdot$

$\cdot N(n+k)] = \underline{\underline{R_{YY}(k) + \sigma_n^2 \delta(k)}}$ $\begin{cases} \frac{5}{3} & k=0 \\ \frac{4}{3} \cdot (\frac{1}{2})^{|k|} & k \neq 0 \end{cases}$

2.3. $R_{XY}(k) = E[X(n)Y(n+k)] =$

$= E[(Y(n)+N(n))Y(n+k)] = E[Y(n)Y(n+k)] = \underline{\underline{R_{YY}(k)}}$

2.4. $\hat{Y}(n) = \sum_{m=0}^{2} h(m)X(n-m)$

$E[\underbrace{(Y(n) - \sum_m h(m)X(n-m))}_{Y(n)-\hat{Y}(n)} X(n-k)] = 0, \quad k=0,1,2$

$E[Y(n)X(n-k) - (\sum_m h(m)X(n-m))X(n-k)] = 0$

$\Rightarrow R_{XY}(k) - \sum h(m) R_{XX}(m-k) = 0 \quad k=0,1,2$

$\Rightarrow R_{XY}(k) = \sum_m h(m)[R_{YY}(m-k) + \sigma_n^2 \delta(m-k)]$

$R_{XY}(0) = \sum_m h(m)(R_{YY}(m) + \sigma_n^2 \delta(m))$

$R_{XY}(1) = \sum_m h(m)(R_{YY}(m-1) + \sigma_n^2 \delta(m-1))$

$\vdots$

$$\begin{bmatrix} R_{YY}(0) \\ R_{YY}(1) \\ R_{YY}(2) \end{bmatrix} = \begin{bmatrix} R_{YY}(0)+\sigma_n^2 & R_{YY}(1) & R_{YY}(2) \\ R_{YY}(1) & R_{YY}(0)+\sigma_n^2 & R_{YY}(1) \\ R_{YY}(-2) & R_{YY}(-1) & R_{YY}(0)+\sigma_n^2 \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$$

$$\begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{5}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{5}{3} \end{bmatrix} \begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$$

$\Updownarrow$

2.5. $\underline{R_{XY}} = \underset{\sim}{R_{XX}} \cdot \underline{h} \Rightarrow \underline{h} = \underset{\sim}{R_{XX}}^{-1} \cdot \underline{R_{XY}} = [\text{hint}] \cdot \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$

$\Rightarrow h = \begin{bmatrix} 0{,}761 \\ 0{,}091 \\ 0{,}011 \end{bmatrix}$

2.6. $E\left[\left|Y(n)-\hat{Y}(n)\right|^2\right] = R_{YY}(0) - \sum_{m=0}^{2} h(m)\, R_{XY}(m)$

$= \frac{4}{3} - 1{,}080 = \underline{\underline{0{,}254}}$

2.7. $\underline{\underline{H(f)}} = \frac{S_{XY}(f)}{S_{XX}(f)} = \frac{\mathcal{F}\{R_{XY}(k)\}}{\mathcal{F}\{R_{XX}(k)\}} = \frac{\mathcal{F}\{R_{YY}(k)\}}{\mathcal{F}\{R_{XX}(k)\}} =$

$= \frac{\mathcal{F}\left\{\frac{4}{3}\left(\frac{1}{2}\right)^{|k|}\right\}}{\mathcal{F}\left\{\frac{4}{3}\left(\frac{1}{2}\right)^{|k|}\right\} + \sigma_n^2} = \frac{\frac{1}{\left|1-\frac{1}{2}\exp(-j2\pi f)\right|^2}}{\frac{1}{\left|1-\frac{1}{2}\exp(-j2\pi f)\right|^2} + \sigma_n^2} =$

$= \frac{1}{\sigma_n^2 \cdot \left|1-\frac{1}{2}\exp(-j2\pi f)\right|^2 + 1} = \underline{\underline{\frac{1}{\frac{17}{12} - \frac{1}{3}\cos(2\pi f)}}}$