Stochastic Processes II (FP-7.5):

Written Examination

Date and Time: Thursday, Feb. 14, 2002, 11.00-13.00.

Hints: Try to solve as many items in the problems as you can. All items have the same weight. Thus, should you encounter some difficulty in solving one question, then skip it and go to the next.

Problem 1: Detection of a Gaussian random variable in background noise

Let us consider the following hypothesis testing problem:

H_0: Only Gaussian noise W is present

H_1 : A random Gaussian signal X plus Gaussian noise W is present.

More specifically the received signal Y under both hypotheses reads:

$$H_0: Y = W$$
$$H_1: Y = X + W$$

where

A. W is a zero-mean Gaussian random variable with variance $\sigma_W^2 = 1$, i.e. $W \sim \mathrm{N}(0,1)$.

B. X is a zero-mean Gaussian random variable with variance σ_X^2, i.e. $X \sim \mathrm{N}(0, \sigma_X^2)$.

We further assume that W and X are independent.

The signal to noise ration is

$$\eta = \left(\frac{\sigma_X}{\sigma_W}\right)^2 = 3$$

1.1. Find the probability density function of Y under both hypotheses, i.e. $f(y \mid H_0)$ and $f(y \mid H_1)$.

Hint: Notice that the variance of the sum of two independent random variables equals the sum of their individual variances.

1.2. Show that the likelihood ratio $L(y) = \dfrac{f(y \mid H_1)}{f(y \mid H_0)}$ is given by

$$L(y) = \frac{1}{\sqrt{1+\eta}} \exp\left(\frac{1}{2}\frac{\eta}{1+\eta}\left|\frac{y}{\sigma_W}\right|^2\right).$$
$$= \frac{1}{2}\exp\left(\frac{3}{8}|y|^2\right)$$

1.3. Calculate the log-likelihood function $l(y) = \ln\left(\dfrac{f(y \mid H_1)}{f(y \mid H_0)}\right)$.

1.4. Find the MAP decision rules for the two following a priori probability distributions:

$$\text{a) } \mathrm{P}[H_0] = \mathrm{P}[H_1] = \frac{1}{2},$$

$$\text{b) } \mathrm{P}[H_0] = \frac{10}{11},\ \mathrm{P}[H_1] = \frac{1}{11}.$$

Hint: Be aware that $\sqrt{y^2} = |y|$.

1.5. Sketch the decision regions of the above decision rules.

1.6. Sketch the areas under the probability densities $f(y \mid H_0)$ and $f(y \mid H_1)$ that correspond to the false alarm probability P_f and the probability of a miss P_m for the two MAP rules specified in 1.4.

1.7. Calculate P_f and P_m for the two MAP rules specified in 1.4.
Hint: Make use of Table D.1 in Appendix D, p. 631 of [Shanmugan].

1.8. Compute the probability P_e of making a decision error for the two MAP rules specified in 1.4.

1.9. Compute the probability of making a decision error for the MAP rule specified in 1.4.a) when in fact the a-priori probability distribution coincides with that given in 1.4.b). Compare with the results obtained in 1.8.

Problem 1: Unbiased linear estimator of an unknown constant

We consider the problem of estimating a constant c based on N noisy observations

$$X_n = c + W_n,\ n = 1, \ldots, N.$$

Here, $W_1, \ldots, W_N$ are independent Gaussian random variables (noise) with zero-mean and variance $(\sigma_W)^2$.

We focus the attention on linear estimators of the form

$$\hat{C} = \sum_{n=1}^{N} h_n X_n \tag{1}$$

that, in addition, are *unbiased.* Notice that the estimator $\hat{C}$ is *unbiased* if

$$E\left[\hat{C}\right] = c,$$

i.e. the expectation of the estimator equals the constant that it estimates.

Notice that the mean

$$\overline{X} = \frac{1}{N} \sum_{n=1}^{N} X_n$$

is an unbiased estimator of the form (1).

2.1. Show that an estimator of the form (1) is unbiased if, and only if its coefficients satisfy the condition

$$\sum_{n=1}^{N} h_n = 1. \tag{2}$$

2.2. Compute the mean squared error $E\left[\left(\hat{C} - c\right)^2\right]$ of an unbiased estimator of the form (1).

Hint: Notice that because of (2), $\left(\sum_{n=1}^{N} h_n X_n\right) - c = \sum_{n=1}^{N} h_n (X_n - c)$.

2.3. Show that the mean $\overline{X} = \frac{1}{N}\sum_{n=1}^{N} X_n$ is optimum among the unbiased estimators of the form (1), in the sense that it minimizes the mean squared error, i.e. $E\left[\left(\hat{C} - c\right)^2\right]$ is minimum if and only if $\hat{C} = \overline{X}$.

Hint: Because of the constraint (2), the objective function to be minimized for finding the optimum coefficients is

$$J(h_1,\ldots,h_N) = E\left[\left(\hat{C} - c\right)^2\right] - \lambda\left(\sum_{n=1}^{N} h_n - 1\right)$$ (λ is the Lagrange multiplicator).

Show that $J(h_1,\ldots,h_N)$ is minimum when $h_1 = h_2 = \ldots = h_N$. From (2) follows then that $h_1 = h_2 = \ldots = h_N = 1/N$, i.e. that $\hat{C} = \overline{X}$.

Problem 1 : Detection of a Gaussian random variable in background noise

Hypothesis testing problem:

$$H_0 : Y = W \qquad W \sim \mathcal{N}(0, \sigma_w^2)$$

$$H_1 : Y = X + W \qquad X \sim \mathcal{N}(0, \sigma_x^2)$$

X, W independent

1.1 Probability density of Y under H_0 and H_1:

$$f(y \mid H_0) = \frac{1}{\sqrt{2\pi}\,\sigma_w} \exp\left\{ -\frac{1}{2\sigma_w^2} |y|^2 \right\}$$

$$f(y \mid H_1) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2 + \sigma_w^2}} \exp\left\{ -\frac{1}{2}\frac{1}{\sigma_x^2 + \sigma_w^2} |y|^2 \right\}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_w \sqrt{1+\eta}} \exp\left\{ -\frac{1}{2\sigma_w^2 (1+\eta)} |y|^2 \right\}$$

1.2 Likelihood ratio:

$$L(y) = \frac{f(y \mid H_1)}{f(y \mid H_0)}$$

$$= \frac{1}{\sqrt{1+\eta}} \exp\left\{ + \frac{1}{2} \underbrace{\left[\frac{1}{\sigma_w^2} - \frac{1}{\sigma_w^2 (1+\eta)} \right]}_{\frac{1+\eta-1}{\sigma_w^2(1+\eta)} = \frac{1}{\sigma_w^2} \cdot \frac{\eta}{1+\eta}} |y|^2 \right\}$$

$$L(y) = \frac{1}{\sqrt{1+\eta}} \exp\left\{ + \frac{1}{2} \cdot \frac{\eta}{1+\eta} \frac{|y|^2}{\sigma_w^2} \right\}$$

1.3 <u>log likelihood function</u>

$$l(y) = \ln(L(y))$$

$$= \frac{1}{2} \cdot \frac{\eta}{1+\eta} \cdot \frac{|y|^2}{\sigma_w^2} - \frac{1}{2}\ln(1+\eta)$$

$$= \frac{3}{8}|y|^2 - \ln(2)$$

1.4 <u>MAP Test:</u>

$$l(y) \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{P[H_0]}{P[H_1]}$$

$$\frac{1}{2} \cdot \frac{\eta}{1+\eta} \frac{|y|^2}{\sigma_w^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln \frac{P[H_0]}{P[H_1]} + \frac{1}{2}\ln(1+\eta)$$

$$\frac{|y|^2}{\sigma_w^2} \underset{H_0}{\overset{H_1}{\gtrless}} \left(\frac{1+\eta}{\eta}\right)\left[2\ln \frac{P[H_0]}{P[H_1]} + \ln(1+\eta)\right]$$

$$\frac{|y|}{\sigma_w} \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{1+\eta}{\eta} \cdot \ln\left[(1+\eta)\left(\frac{P[H_0]}{P[H_1]}\right)^2\right]} = \tilde{\gamma}$$

$$\sqrt{\frac{4}{3} \cdot \ln\left[4 \cdot \left(\frac{P[H_0]}{P[H_1]}\right)^2\right]}$$

a) $\frac{P[H_0]}{P[H_1]} = 1$:

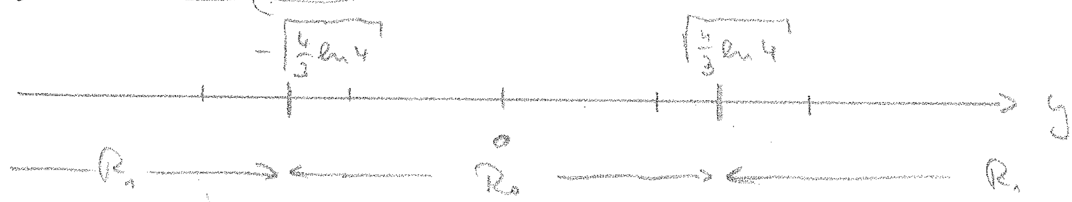
$$\frac{|y|}{\sigma_w} \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{1+\eta}{\eta} \cdot \ln(1+\eta)} = \sqrt{\frac{4}{3} \cdot \ln 4} \approx 1.36$$

b) $\frac{P[H_0]}{P[H_1]} = 10$

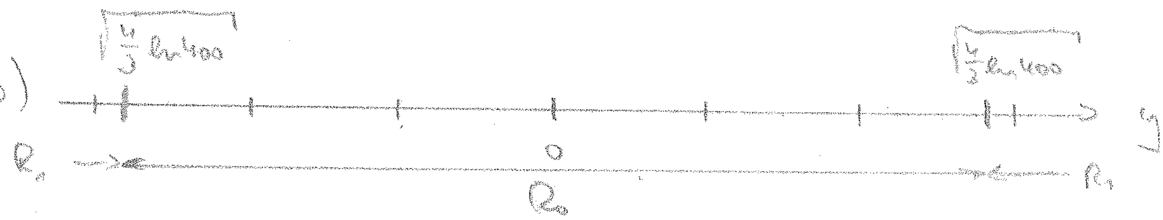
$$\frac{|y|}{\sigma_w} \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{1+\eta}{\eta} \cdot \ln(100(1+\eta))} = \sqrt{\frac{4}{3}\ln(400)} \approx 2.83$$

1.5. Decision regions:

a)

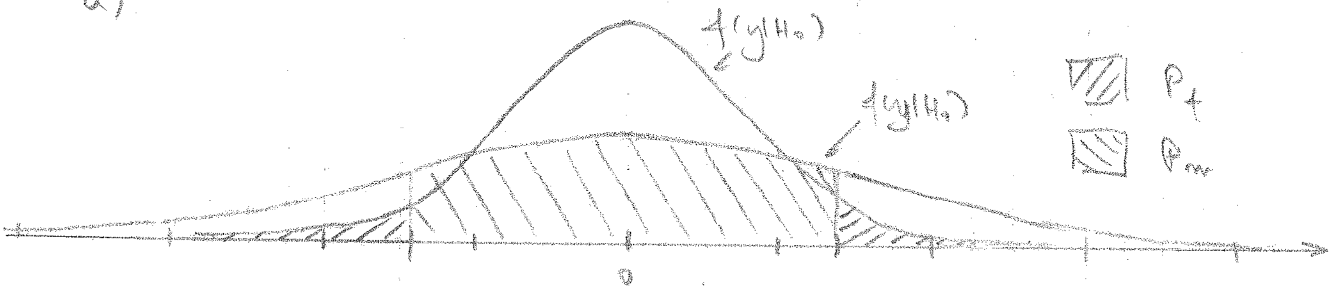


b)



1.6. Sketch of P_f and P_m:

a)



b) Ditto above

1.7. Calculation of P_f and P_m

$$P_f = P\left[\frac{|y|}{\sigma_w} > \tilde{\gamma} \mid H_0\right]$$

$$= P\left[\frac{|w|}{\sigma_w} > \tilde{\gamma}\right]$$

$$= 2Q(\tilde{\gamma})$$

$$P_m = P\left[\frac{|y|}{\sigma_w} < \tilde{\gamma} \mid H_1\right]$$

$$= P\left[\frac{|x+w|}{\sigma_w} < \tilde{\gamma}\right]$$

$$= P\left[\frac{|x+w|}{\sigma_w\sqrt{1+\xi}} < \frac{\tilde{\gamma}}{\sqrt{1+\xi}}\right]$$

$$= 1 - 2Q\left(\frac{\tilde{\gamma}}{\sqrt{1+\xi}}\right)$$

a) $P_f = 2Q\left(\sqrt{\frac{4}{3}\ln 4}\right)$

$= 2Q(1.36) = \ldots$

$P_m = 1 - 2Q\left(\frac{\sqrt{\frac{4}{3}\ln 4}}{\sqrt{4}}\right) = 1 - 2Q\left(\sqrt{\frac{\ln 4}{3}}\right)$

$= 1 - 2Q(0.68)$

b) $P_f = 2Q\left(\sqrt{\frac{4}{3}\ln 400}\right)$

$= 2Q(2.83)$

$P_m = 1 - 2Q\left(\sqrt{\frac{1}{3}\ln 400}\right)$

$= 1 - 2Q(1.41)$

1.8. Error probability

$P_e = P[H_0] \cdot P_f + P[H_1] \cdot P_m$

a) $P_e = \frac{1}{2}(P_f + P_m)$

b) $P_e = \frac{1}{11}\left[10 P_f + P_m\right]$

1.9.

$P_e = \frac{10}{11} \cdot [P_f \text{ in } 18a)] + \frac{1}{11}[P_m \text{ in } 18a)] \geq P_e \text{ in } 18b)$

Problem 2: Unbiased linear estimator of an unknown constant

2.1: Unbiasedness:

$$E[\hat{C}] = E\left[\sum_{n=1}^{N} h_n X_n\right]$$
$$= E\left[\sum_{n=1}^{N} h_n (c + W_n)\right]$$
$$= \left(\sum_{n=1}^{N} h_n\right) c + \sum_{n=1}^{N} h_n \underbrace{E[W_n]}_{=0}$$
$$= c \quad \text{if, and only if} \quad \sum_{n=1}^{N} h_n = 1.$$

2.2 Mean squared error

$$E[(\hat{c} - c)^2] = E\left[\left(\sum_{n=1}^{N} h_n X_n - c\right)^2\right]$$
$$= E\left[\left(\sum_{n=1}^{N} h_n (X_n - c)\right)^2\right]$$
$$= E\left[\left(\sum_{n=1}^{N} h_n W_n\right)^2\right]$$
$$= \sum_{n=1}^{N} h_n^2 \cdot \sigma_w^2$$
$$= \sigma_w^2 \sum_{n=1}^{N} h_n^2$$

2.3 Optimum unbiased estimator of the form (1):

$$\frac{\partial}{\partial h_n} J(h_1, \dots, h_N) = \frac{\partial}{\partial h_n}\left[\sigma_w^2 \sum_{n'=1}^{N} h_{n'}^2 - \lambda\left(\sum_{n'=1}^{N} h_{n'} - 1\right)\right]$$
$$= 2\sigma_w^2 h_n - \lambda$$

$$\frac{\partial J}{\partial h_n} = 0 \iff h_n = \frac{\lambda}{2\sigma_w^2} \quad \text{for any } n = 1, \dots, N$$

$$\Rightarrow h_1 = h_2 = \dots = h_N \overset{(1)}{=} 1/N.$$