Stochastic Processes – Part II (FP-7.5)

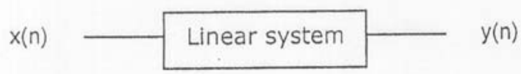
Written Examination OMPRØVE

Date and Time: Friday, Feb. 14, 2003, 11.00-13.00.

Hints: Try to solve as many items in the problem as you can. All items have the same weight. Thus, should you encounter some difficulty in solving one question, then skip it and go to the next.

Problem 1: Linear systems and WSS processes

Let us consider the non-causal linear system,



the input-output relationship of which is given by

$$y(n) = \frac{1}{\sqrt{2}}x(n) - \frac{1}{\sqrt{2}}x(n-4).$$

1.1. Derive and plot the impulse response $h(n)$ of the linear system.

1.2. Derive the transfer function $H(f)$ of the linear system.

1.3. Compute and plot the amplitude spectrum $|H(f)|$ of the linear system.

1.4. Compute the autocorrelation function $R_{hh}(k) = \sum_n h(n)h(n+k)$ of the impulse response $h(n)$.

1.5. Compute the Fourier transform of $R_{hh}(k)$.

Let us now assume that the input sequence is a zero-mean WSS process $\{X(n)\}$ with autocorrelation function,

$$R_{XX}(k) = \left(\frac{4}{5}\right)^{|k|}$$

1.6. Compute the variance of $\{X(n)\}$.

1.7. Compute the power spectrum $S_{XX}(f)$ of $\{X(n)\}$.
Hint: Compare $R_{xx}(k)$ with the autocorrelation of an AR(1) process.

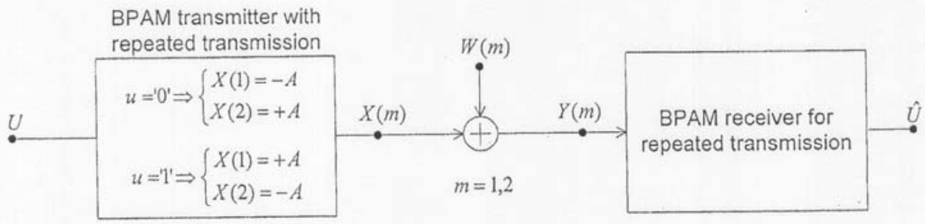
1.8. Derive the autocorrelation function $R_{YY}(k) = \mathbf{E}[Y(n)Y(n+k)]$ of the output process $\{Y(n)\}$.

1.9. Derive and plot the power spectrum $S_{YY}(f)$ of $\{Y(n)\}$.

1.10. Compute the variance of $\{Y(n)\}$.

Problem 2: Detection of BPAM signals corrupted by additive white Gaussian noise

Let us consider the following simplified model of a BPAM (Binary Pulse Amplitude Modulation) system transmitting **two** symbols per information bit across an additive white Gaussian channel.



The signals $X(1), X(2)$ at the transmitter output depend on the input bit U as follows:

$$[X(1), X(2)]^{\mathrm{T}} = \begin{cases}[-A,+A]^{\mathrm{T}}; & U='0'\\ [+A,-A]^{\mathrm{T}}; & U='1'\end{cases},$$

where $A>0$.

The received signals $Y(1)$ and $Y(2)$ read

$$Y(m) = X(m) + W(m),\ m = 1,2.$$

Here, $W(1)$ and $W(2)$ are zero-mean Gaussian random variables with identical variance $\sigma_W^2 = \mathbf{E}\left[W(1)^2\right] = \mathbf{E}\left[W(2)^2\right]$.

The signal-to-noise ratio $\eta = \frac{\mathbf{E}\left[X(m)^2\right]}{\mathbf{E}\left[W(m)^2\right]} = \left(\frac{A}{\sigma_W}\right)^2$ is equal to 10 dB, i.e. $\eta = 10$.

2.1. Formulate the detection process in the BPAM receiver as a binary hypothesis testing problem where $\mathbf{Y} = [Y(1), Y(2)]^{\mathrm{T}}$ is the observation **vector** and the null hypothesis is $H_0 \equiv ["U='1'"]$, i.e. that the bit at the input of the BPAM transmitter is a '1'.

2.2. Find the probability density function of $\mathbf{Y}$ under both hypotheses, i.e. $f(\mathbf{y} \mid H_0)$ and $f(\mathbf{y} \mid H_1)$.

2.3. Calculate the log-likelihood ratio $\ell(\mathbf{y}) = \ln\left(\frac{f(\mathbf{y} \mid H_1)}{f(\mathbf{y} \mid H_0)}\right)$.

2.4. Find the MAP decision rules for the two following a priori probability distributions:

a) $\mathbf{P}[H_0] = \mathbf{P}[H_1] = \frac{1}{2}$,

b) $\mathbf{P}[H_0] = \frac{\exp(3)}{1+\exp(3)} \approx 0.9526$, $\mathbf{P}[H_1] = \frac{1}{1+\exp(3)} \approx 0.0474$.

Hint: Express the decision rule as $y(1) - y(2) \geq \gamma$ for some γ to be calculated.

2.5. Sketch the decision regions for the two decision rules in Item 4.

2.6. Show that the probability density functions of the sum $Z = Y(1) - Y(2)$ under both hypotheses are given by

$$f(z \mid H_0) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{2}\sigma_W} \exp\left\{-\frac{1}{4\sigma_W^2}(z-2A)^2\right\},$$

$$f(z \mid H_1) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{2}\sigma_W} \exp\left\{-\frac{1}{4\sigma_W^2}(z+2A)^2\right\}.$$

2.7. Sketch the areas under the graphs of $f(z \mid H_0)$ and $f(z \mid H_1)$ that correspond to the false alarm probability P_f and the probability of a miss P_m for the two MAP rules specified in Item 4.

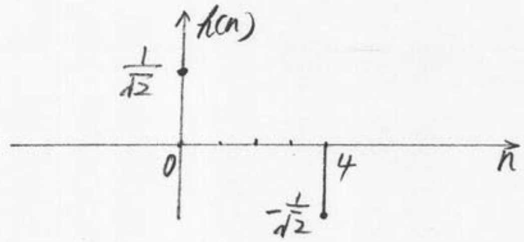
2.8. Calculate P_f and P_m for the two MAP rules specified in Item 4.
Hint: Make use of Table D.1 in Appendix D, p.631 of [Shanmugan] or the subsequent table.

2.9. Compute the probability P_e of making a decision error (bit-error probability) for the two MAP rules specified in Item 4.

2.10. Compute P_e for the first MAP rule specified in Item 4 when in fact the a-priori probability distribution coincides with that given in Item 4.b. Compare with the results obtained in Item 9.

y	$Q(y)$
4.00	3.1671×10^{-5}
4.10	2.0658×10^{-5}
4.20	1.3346×10^{-5}
4.30	8.5399×10^{-6}
4.40	5.4125×10^{-6}
4.50	3.3977×10^{-6}
4.60	2.1125×10^{-6}
4.70	1.3008×10^{-6}
4.80	7.9333×10^{-7}
4.90	4.7918×10^{-7}
5.00	2.8665×10^{-7}

1.1 $h(n) = \frac{1}{\sqrt{2}}\delta(n) - \frac{1}{\sqrt{2}}\delta(n-4)$



1.2 $H(f) = \mathcal{F}\{h(n)\}$

$= \mathcal{F}\{\frac{1}{\sqrt{2}}[\delta(n) - \delta(n-4)]\}$

$= \frac{1}{\sqrt{2}}[\sum_n \delta(n)\exp(-j2\pi f n) - \sum_n \delta(n-4)\exp(-j2\pi f n)]$

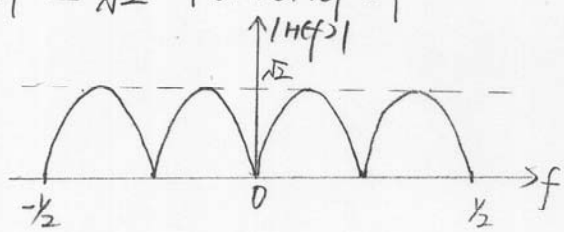
$= \frac{1}{\sqrt{2}}\{1 - \exp(-j8\pi f)\}$

$= \frac{1}{\sqrt{2}}\exp(-j4\pi f)\cdot[\exp(j4\pi f) - \exp(-j4\pi f)]$

$= \frac{1}{\sqrt{2}}\exp(-j4\pi f)\cdot j\cdot 2\sin(4\pi f)$

$= \sqrt{2}\exp(-j4\pi f)\cdot\sin(4\pi f)\cdot j$

1.3 $|H(f)| = \sqrt{2}\cdot|\sin(4\pi f)|$



1.4 $R_{hh}(k) = \sum_n h(n)h(n+k)$

$= \sum_n (\frac{1}{\sqrt{2}}\delta(n) - \frac{1}{\sqrt{2}}\delta(n-4))(\frac{1}{\sqrt{2}}\delta(n+k) - \frac{1}{\sqrt{2}}\delta(n+k-4))$

$= \frac{1}{2}\sum_n[\delta(n)\delta(n+k) - \delta(n)\delta(n+k-4) - \delta(n-4)\delta(n+k)$
$+ \delta(n-4)\delta(n+k-4)]$

$= \frac{1}{2}\cdot[\delta(k) - \delta(k-4) - \delta(k+4) + \delta(k)]$

$= -\frac{1}{2}\delta(k-4) + \delta(k) - \frac{1}{2}\delta(k+4)$

1.5 $S_{hh}(f) = \mathcal{F}\{R_{hh}(k)\}$

$= -\frac{1}{2}\cdot[\exp(-j8\pi f) - 2 + \exp(j8\pi f)]$

$= -\frac{1}{2}\cdot[2\cos(8\pi f) - 2]$

$= 1 - \cos(8\pi f)$

1.6 $R_{xx}(0) = (\frac{4}{5})^0 = 1$

1.7 $S_{xx}(f) = \mathcal{F}\{ R_{xx}(k) \}$

$$= \sum_k R_{xx}(k) \cdot \exp(-j2\pi f k)$$

$$= \sum_k \cdot (\tfrac{4}{5})^{|k|} \cdot \exp(-j2\pi f k)$$

$$= \frac{1-(\frac{4}{5})^2}{|1-(\frac{4}{5})^2 \cdot \exp(-j2\pi f)|^2}$$

1.8 $R_{YY}(k) = E[Y(n)\, Y(n+k)]$

$$= E\left[\left(\tfrac{1}{\sqrt{2}} x(n) - \tfrac{1}{\sqrt{2}} x(n-4)\right) \cdot \left(\tfrac{1}{\sqrt{2}} x(n+k) - \tfrac{1}{\sqrt{2}} x(n+k-4)\right)\right]$$

$$= \tfrac{1}{2} \cdot (R_{xx}(k) - R_{xx}(k-4) - R_{xx}(k+4) + R_{xx}(k))]$$

$$= R_{xx}(k) - \tfrac{1}{2} R_{xx}(k-4) - \tfrac{1}{2} R_{xx}(k+4)$$

1.9

$$S_{YY}(f) = |H(f)|^2 \cdot S_{xx}(f)$$

$$= 2 \cdot |\sin(4\pi f)|^2 \cdot \frac{1-(\frac{4}{5})^2}{|1-(\frac{4}{5})^2 \cdot \exp(-j2\pi f)|^2}$$

$$= \frac{18 \cdot |\sin(4\pi f)|^2}{|5 - \frac{16}{5} \cdot \exp(-j2\pi f)|^2}$$

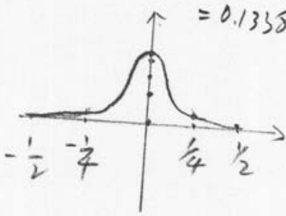
$f = 0$: $S_{YY}(0) = 2\frac{7}{9} = 2.\dot{7}$

$f = \frac{1}{4}$: $S_{YY}(\frac{1}{4}) = \frac{225}{881}$

$= 0.2554$

$f = \frac{1}{2}$: $S_{YY}(\frac{1}{2}) = \frac{225}{1681}$

$= 0.1338$



1.10. $Var(Y(n)) = R_{YY}(k)\big|_{k=0}$

$$= R_{xx}(0) - \tfrac{1}{2} R_{xx}(-4) - \tfrac{1}{2} R_{xx}(4)$$

$$= 1 - \tfrac{1}{2} \cdot (\tfrac{4}{5})^4 - \tfrac{1}{2} \cdot (\tfrac{4}{5})^4$$

$$= 1 - (\tfrac{4}{5})^4$$

$$= 1 - \frac{256}{625} = \frac{369}{625}$$

2.1 :

$$H_0: \begin{bmatrix} Y(1) \\ Y(2) \end{bmatrix} = \begin{bmatrix} +A \\ -A \end{bmatrix} + \begin{bmatrix} W(1) \\ W(2) \end{bmatrix}$$

$$H_1: \begin{bmatrix} Y(1) \\ Y(2) \end{bmatrix} = \begin{bmatrix} -A \\ +A \end{bmatrix} + \begin{bmatrix} W(1) \\ W(2) \end{bmatrix}$$

2.2

$$S_0 = [+A \ -A]^T$$
$$S_1 = [-A \ +A]^T$$

$$f(\underline{y}|H_0) = \frac{1}{2\pi\sigma_w^2} \exp\left\{-\frac{1}{2\sigma_w^2}\cdot\left[(y(1)-A)^2 + (y(2)+A)^2\right]\right\}$$

$$f(\underline{y}|H_1) = \frac{1}{2\pi\sigma_w^2} \exp\left\{-\frac{1}{2\sigma_w^2}\cdot\left[(y(1)+A)^2 + (y(2)-A)^2\right]\right\}$$

2.3.

$$l(\underline{y}) = \ln\frac{f(\underline{y}|H_1)}{f(\underline{y}|H_0)} = -\frac{1}{2\sigma_w^2}\Big[(y(1)+A)^2 + (y(2)-A)^2 - (y(1)-A)^2 - (y(2)+A)^2\Big]$$

$$= -\frac{1}{2\sigma_w^2}\cdot\Big[y^2(1) + y^2(2) - y^2(1) - y^2(2) + 2y(1)A - 2y(2)A + 2y(1)A - 2y(2)A\Big]$$

$$= -\frac{1}{2\sigma_w^2}\left(4y(1)A - 4y(2)A\right)$$

$$= -\frac{2A}{\sigma_w^2}\left[y(1) - y(2)\right]$$

2.4

$$l(y) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma.$$

a) $\gamma = \ln\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = 0$

$$-\frac{2A}{\sigma_w^2}\cdot(y(1)-y(2)) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$y(1)-y(2) \underset{H_1}{\overset{H_0}{\gtrless}} 0$$

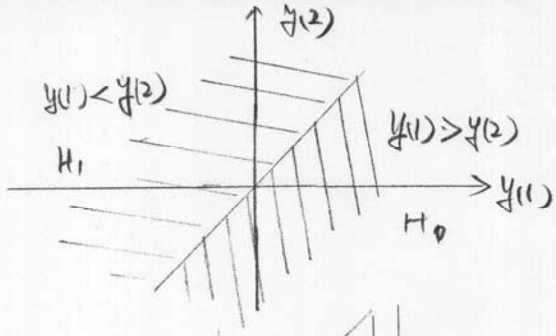
b) $\gamma = \ln(e^3) = 3$

$$-\frac{2A}{\sigma_w^2}(y(1)-y(2)) \underset{H_0}{\overset{H_1}{\gtrless}} 3$$

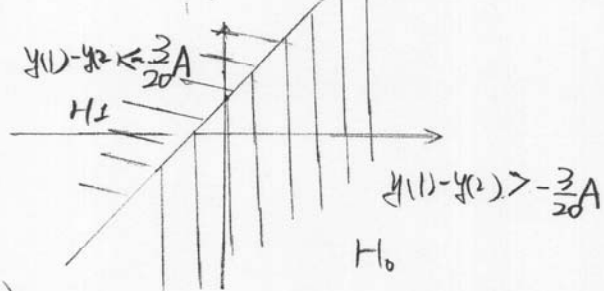
$$y(1)-y(2) \underset{H_1}{\overset{H_0}{\gtrless}} -\frac{3\sigma_w^2}{2A} = -\frac{3\cdot\frac{A^2}{10}}{2\cdot A} = -\frac{3}{20}A$$

2.5 a)

$y(1) - y(2) = 0$



b)



2.6. $Z = Y(1) - Y(2)$

H_0: $E[Z] = A + A = 2A$, $Var[Z] = 2\sigma_w^2$

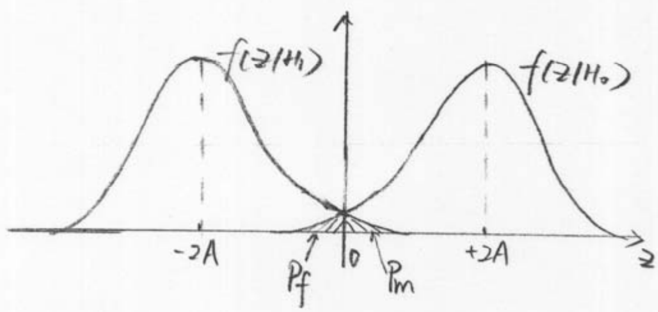
H_1: $E[Z] = -A - A = -2A$, $Var[Z] = 2\sigma_w^2$

$$f(Z/H_0) = \frac{1}{\sqrt{2\pi}\sqrt{2\sigma_w^2}} \cdot \exp\left\{-\frac{1}{2 \cdot 2\sigma_w^2} \cdot (Z - 2A)^2\right\}$$

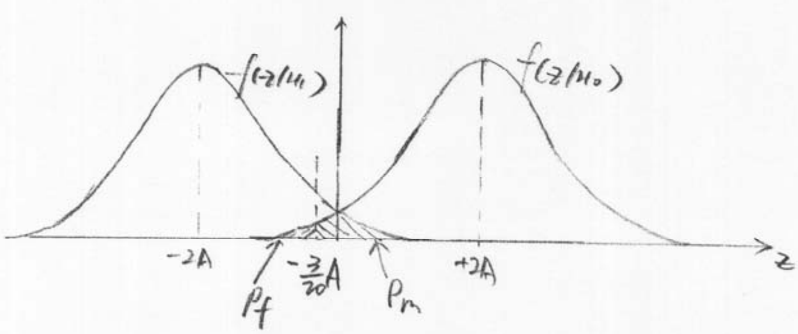
$$f(Z/H_1) = \frac{1}{\sqrt{2\pi}\sqrt{2\sigma_w^2}} \cdot \exp\left\{-\frac{1}{2 \cdot 2\sigma_w^2} (Z + 2A)^2\right\}.$$

2.7.

a)



b)



2.8 a) $P_f = P_m = \int_0^{+\infty} f(z|H_1)\,dz$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}\,\sigma_w} \int_0^{+\infty} \exp\left(-\frac{1}{4\sigma_w^2}(z+2A)^2\right) dz$$

let $u = \frac{z+2A}{\sqrt{2}\,\sigma_w} \Rightarrow dz = \sqrt{2}\,\sigma_w\,du$

$z = 0 \Rightarrow u = \sqrt{2}\frac{A}{\sigma_w} = \sqrt{20}$

$$= \frac{1}{\sqrt{2\pi}} \int_{\sqrt{20}}^{+\infty} \exp\left(-\frac{1}{2}u^2\right) du$$

$= Q(\sqrt{20})$

$= Q(4.4721)$

$= 3.3977 \times 10^{-6}$

b) $P_f = \int_{-\infty}^{-\frac{3}{20}A} f(z|H)\,dz$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}\,\sigma_w} \int_{-\infty}^{-\frac{3}{20}A} \exp\left(-\frac{1}{4\sigma_w^2}(z-2A)^2\right) dz$$

let $u = \frac{z-2A}{\sqrt{2}\,\sigma_w} \Rightarrow dz = \sqrt{2}\,\sigma_w\,du$

$z = -\frac{3}{20}A \Rightarrow u = \frac{-\frac{3}{20}-2}{\sqrt{2}\,\sigma_w}A = \frac{-\frac{43}{20}}{\sqrt{2}}\cdot\sqrt{10} = -\frac{43}{20}\sqrt{5}$

$= -4.8075$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-4.8075} \exp\left(-\frac{1}{2}u^2\right) du$$

$= Q(4.8075)$

$= 7.6415 \times 10^{-7}$

$P_m = \int_{-\frac{3}{20}A}^{+\infty} f(z|H_1)\,dz$

$$= \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}\cdot\sigma_w} \int_{-\frac{3}{20}A}^{+\infty} \exp\left(-\frac{1}{4\sigma_w^2}(z+2A)^2\right) dz$$

let $u = \frac{z+2A}{\sqrt{2}\,\sigma_w} \Rightarrow dz = \sqrt{2}\,\sigma_w\,du$

$z = -\frac{3}{20}A \Rightarrow u = \frac{-\frac{37}{20}A}{\sqrt{2}\,\sigma_w} = -\frac{37}{20}\sqrt{5}$

$= -4.1367$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-4.1367}^{+\infty} \exp\left(-\frac{1}{2}u^2\right) du$$

$= Q(4.1367)$

$= 1.7617 \times 10^{-5}$

2.9 a) $P_e = P_f \cdot P[H_0] + P_m \cdot P[H_1]$

$= P_f = 3.3977 \times 10^{-6}$

b) $P_e = P_f\,P[H_0] + P_m \cdot P[H_1]$

$= 7.6415 \times 10^{-7} \times 0.9526 + 1.7617 \times 10^{-5} \times 0.474$

$= 1.5630 \times 10^{-6}$ —— (1)

2.10. $\tilde{P}_e = P_f \cdot P[H_0] + P_m \cdot [H_1]$

$= 3.3977 \times 10^{-6}$ —— (2)

(2) > (1).