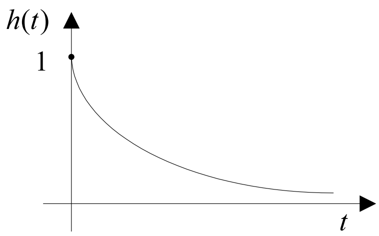
Stochastic Processes II (FP-7.5)
Solution Set 2

Problem 2.1 (Problem 4.14 in Shanmugan)

Solution:

- Impulse response:

$$h(t) = \begin{cases} \exp(-t), & t \geq 0 \\ 0, & \text{elsewhere} \end{cases}$$



- Transfer function:

$$\begin{aligned}
H(f) &= \int_{-\infty}^{+\infty} h(t) \exp(-j2\pi f t) dt \\
&= \int_{0}^{+\infty} \exp(-t) \exp(-j2\pi f t) dt \\
&= \int_{0}^{+\infty} \exp(-[1 + j2\pi f]t) dt \\
&= \frac{1}{-[1 + j2\pi f]} \exp(-[1 + j2\pi f]t)|_0^{+\infty} \\
&= -\frac{1}{1 + j2\pi f}[0 - 1] \\
H(f) &= \frac{1}{1 + j2\pi f}
\end{aligned}$$

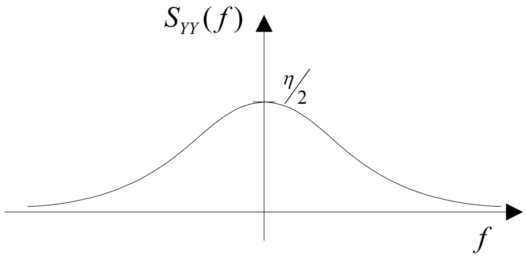
- Power Spectrum of $Y(t)$:

  $X(t)$ is a white noise with unit variance

  ◇ $E[X(t)] = 0 \Rightarrow E[Y(t)] = 0$

  ◇ $S_{XX}(f) = \frac{\eta}{2} \Rightarrow S_{YY}(f) = \frac{\eta}{2} \frac{1}{|1+j2\pi f|^2}$

$$S_{YY}(f) = \frac{\eta}{2} \frac{1}{1 + (2\pi f)^2}$$



- Variance of $Y(t)$:

$$\begin{aligned}
E[Y^2(t)] &= R_{YY}(0) \\
&= (R_{hh}(\tau) * \underbrace{R_{XX}(\tau)}_{=\frac{\eta}{2}\delta(\tau)})|_{\tau=0} \\
&= \frac{\eta}{2} R_{hh}(0) \\
&= \frac{\eta}{2} \int_{-\infty}^{+\infty} h(t)^2 dt \\
&= \frac{\eta}{2} \int_{0}^{+\infty} \exp(-2t) \\
&= \frac{\eta}{2} \cdot \frac{1}{-2} \exp(-2t)|_0^{\infty} \\
&= -\frac{\eta}{4}[0-1] \\
E[Y^2(t)] &= \frac{\eta}{4}
\end{aligned}$$

Problem 2.2 (Problem 5.3 in Shanmugan)

Solution:

- Mean:

$$E[Y(n)] = \frac{1}{2}E[Y(n-1)] + \underbrace{E[Z(n)]}_{=0} \tag{1}$$

If $Y(n)$ is WSS,

$$E[Y(n)] = E[Y(n-1)] = \mu_Y.$$

Inserting the above equation in (1) yields

$$\mu_Y = \frac{1}{2}\mu_Y.$$

The only possibility is that

$$E[Y(0)] = \mu_Y = 0.$$

- Variance:

$$\begin{aligned}\mathrm{Var}[Y(n)] &= E[Y^2(n)] \\ &= E[(\frac{1}{2}Y(n-1)+Z(n))^2] \\ &= \frac{1}{4}\underbrace{E[Y^2(n-1)]}_{=\mathrm{Var}[Y(n-1)]}+\underbrace{E[Y(n-1)Z(n)]}_{=0}+\underbrace{E[Z^2(n)]}_{=\sigma_Z^2} \end{aligned} \quad (2)$$

If $Y(n)$ is WSS,

$$\mathrm{Var}[Y(n)] = \mathrm{Var}[Y(n-1)] = \sigma_Y^2.$$

Inserting the above equation in (2) yields

$$\sigma_Y^2 = \frac{1}{4}\sigma_Y^2 + \sigma_Z^2.$$

Solving, we obtain

$$\frac{3}{4}\sigma_Y^2 = \sigma_Z^2 \quad \Rightarrow \quad \sigma_Y^2 = E[Y^2(0)] = \frac{4}{3}\sigma_Z^2.$$

Problem 2.3 (Problem 5.4 in Shanmugan)

Solution:

(a) $\mu_Y = 0$, according to the solution for Problem 5.3;

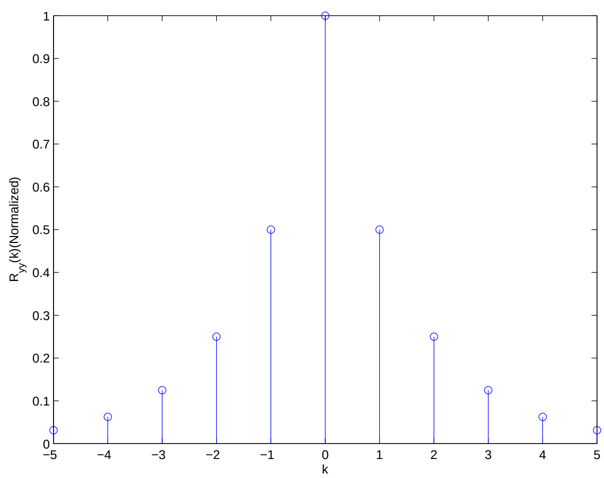
(b) $\sigma_Y^2 = \frac{4}{3}\sigma_Z^2$, according to the solution for Problem 5.3;

(c) Autocorrelation function

$$R_{YY}(k) = E[Y(n)Y(n+k)]$$

$$\begin{aligned}
k=0: \quad R_{YY}(0) &= \sigma_Y^2 = \frac{4}{3}\sigma_Z^2 \\
k=1: \quad R_{YY}(1) &= E[Y(n)Y(n+1)] \\
&= E[Y(n)[\frac{1}{2}Y(n) + Z(n+1)]] \\
&= \frac{1}{2}E[Y^2(n)] + \underbrace{E[Y(n)Z(n+1)]}_{=0} \\
&= \frac{1}{2}R_{YY}(0) \\
&= \frac{1}{2}\sigma_Y^2 \\
k=2: \quad R_{YY}(2) &= E[Y(n)Y(n+2)] \\
&= E[Y(n)[\frac{1}{2}Y(n+1) + Z(n+2)]] \\
&= \frac{1}{2}E[Y(n)Y(n+1)] + \underbrace{E[Y(n)Z(n+2)]}_{=0} \\
&= \frac{1}{2}R_{YY}(1) \\
&\cdots \\
R_{YY}(k) &= (\frac{1}{2})^{|k|}R_{YY}(0)
\end{aligned}$$

$$\text{In summary,} \quad R_{YY}(k) = (\frac{1}{2})^{|k|}\frac{4}{3}\sigma_Z^2.$$



(d) Autocorrelation coefficient $r_{YY}(k)$

$$
\begin{aligned}
r_{YY}(k) &= \frac{R_{YY}(k)}{R_{YY}(0)} \\
&= \left(\frac{1}{2}\right)^{|k|}.
\end{aligned}
$$

(e) Power spectrum

$$
\begin{aligned}
S_{YY}(f) &= \frac{\sigma_Z^2}{\left|1 - \frac{1}{2}\exp(-j2\pi f)\right|^2} \\
&= \frac{\sigma_Z^2}{[1 - \frac{1}{2}\exp(-j2\pi f)](1 - \frac{1}{2}\exp(j2\pi f))} \\
S_{YY}(f) &= \frac{\sigma_Z^2}{\frac{5}{4} - \cos(2\pi f)}
\end{aligned}
$$