Stochastic Analysis for Engineers

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Fall 2004, Aalborg University Esbjerg

Course Web:**http://www.cs.aue.auc.dk/~yang/course/stoc04.htm**

Objective

- To give students an **understanding** of the description of stochastic signals in order to perform filtering and detection
- To enable students to **apply** estimation and detection methods for simple problems in connection with stationary stochastic processes
- To give students an **understanding** of spectral estimation techniques

Textbook

- K. Sam Shanmugan and BArthur M. reipohl: "***Random Signals - Detection, Estimation and Data Analysis*** ", John Wiley Sons, Inc., 1988.

Content

- Discrete linear models
  - Autoregressive (AR) processes
  - Moving average (MA) models
  - Autoregressive moving average (ARMA) models
- Signal detection
  - Hypothesis testing
  - Decision theory and decision rules
  - Binary detection
  - M-ary detection
- Linear minimum mean-square error estimation
  - Linear minimum mean-square error estimators
  - Nonlinear minimum mean-square error estimators
  - Joint Gaussian random variables

Content (cont'd)

- Discrete time Wiener filters
  - Non-causal discrete time Wiener filters
  - Causal discrete time Wiener filters

- Discrete time Kalman filters
  - Kalman filters
  - Extended Kalman filters

- Parameter estimation of stochastic processes
  - Model-free estimation: mean value, autocorrelation, PSD
  - Model-based estimation for AR, MA, ARMA processes

- Case studies

Ten Lecture Topics

Topics	Sections in Shanmugan
Response of linear systems to random inputs	4.1, 4.2, 4.3
Discrete linear stochastic models	5.2
Detection of known signals (Part 1)	6.2, 6.3, 6.5
Detection of known signals (Part 2)	6.2, 6.3, 6.5
Mean-square error filtering and estimation, Wiener filters (Part 1)	7.2, 7.3 7.5
Wiener filters (Part 2)	7.5, 7.7
Kalman filters (Part 1)	7.6
Kalman filters (Part 2)	7.6
Model-free and spectral estimation (Part 1)	9.3, 9.4
Model-free and spectral estimation (Part 2)	9.3, 9.4

What have we learned in Sem6?

- **MM1:** Definition and description of stochastic processes
- **MM2:** Special classes of stochastic processes and stationarity
- **MM3:** Autocorrelation and power spectral density functions of WSS processes
- **MM4:** Continuity, differentiation, integration, time averaging and ergodicity
- **MM5:** Response of Linear Systems to Random Signals

Random Processes

- Let **S** be the sample space of a random experiment and let **t** be a variable that can have values in the set $\Gamma \subset \mathbf{R}_1$, the real line. A **real-valued random process X(t), t** $\in \Gamma$, is a measurable function on $\Gamma \times \mathbf{S}$ that maps $\Gamma \times \mathbf{S}$ onto $\mathbf{R}_1$

- A real-valued random process can be described by its nth order distribution function like

$$F_{X(t_1),X(t_2),\ldots X(t_n)}(x_1, x_2, \ldots x_n) = P[X(t_1) \le x_1, \ldots, X(t_n) \le x_n]$$

for all **n** and $\mathbf{t_1, \ldots, t_n} \in \Gamma$

Method of Description

- First and second order characteristics
  - The mean of **X(t)**

$$\mu_X(t) = E\{X(t)\}$$

  - The autocorrelation of **X(t)**
  - $$R_{XX}(t_1,t_2) = E\{X^*(t_1)X(t_2)\}$$
  - The autocovariance of **X(t)**

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X{}^*(t_1)\mu_X(t_2)\}$$

  - The correlation coefficient of **X(t)**

$$r_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_1)C_{XX}(t_2,t_2)}}$$

Matlab Calculations

- **randn(m,n)** function generates arrays of random numbers whose elements are normally distributed with mean 0, and variance 1
- **M = mean(A)** returns the mean values of the elements along different dimensions of an array.
- **C = cov(x)** where x is a vector returns the variance of the vector elements.
- **C=xcorr(X,Y)** estimates the cross-correlation sequence of a random process. Autocorrelation is handled as a special case.
- **C=corrcoef(X)** returns a matrix of correlation coefficients calculated from an input matrix whose rows are observations and whose columns are variables.

Strict-Sense Stationarity (SSS)

- A random process **X(t)** is called **time stationary** or **stationary in the strict sense (SSS)** if all of the distribution functions describing the process are invariant under a time translation, i.e.,

  for all $t_1, t_2, \ldots t_k, t_1+\tau, t_2+\tau, \ldots, t_k+\tau \in \Gamma$, and all $k=1,2,\ldots,$

$$P[X(t_1)\leq x_1, X(t_2)\leq x_2, \ldots, X(t_k)\leq x_k]$$
$$=P[X(t_1+\tau)\leq x_1, X(t_2+\tau)\leq x_2, \ldots, X(t_k+\tau)\leq x_k]$$

Wide-Sense Stationarity (WSS)

- A random process **X(t)** is said to be **stationary in the wide sense (WSS)** if its mean is a constant and the correlation function depends only on the time difference, i.e.,

$$E\{X(t)\}=\mu_X=\text{constant},$$

$$E\{X^*(t)X(t+\tau)\}=R_{XX}(\tau)$$

Autocorrelation Properties

$$R_{XX}(\tau)=E\{X(t)X(t+\tau)\}$$

- $R_{XX}(0)=E\{X^2(t)\}\geq 0$ - **average power**
- $R_{XX}(\tau)$ is an even function of τ, i.e., $R_{XX}(\tau)= R_{XX}(-\tau)$
- $R_{XX}(\tau)$ is bounded by $R_{XX}(0)$, i.e., $|R_{XX}(\tau)|\leq R_{XX}(0)$
- If $X(t)$ contains a periodic component, then $R_{XX}(\tau)$ will also contain a periodic component
- If $\lim_{\tau\to\infty} R_{XX}(\tau)=C$, then $C= \mu_X^2$
- If $R_{XX}(T_0)= R_{XX}(0)$ for some nonzero T_0, then $R_{XX}(\tau)$ is periodic with a period T_0
- If $R_{XX}(0)<\infty$, and $R_{XX}(\tau)$ is continuous at $\tau=0$, then it is continuous for every τ

Frequency Analysis of Random Signals

How about the spectral properties of the random processes?

- Direct Fourier transform can not be applied to random signals
- The autocorrelation function $\mathbf{R_{XX}(\tau)}$ contains some information about the frequency of the random signals
- Consider real WSS random processes…

Power Spectral Density Function

- For the random WSS random process **X(t)**, the **PSD function** is defined as

$$S_{XX}(f) = F\{R_{XX}(\tau)\} = \int_{-\infty}^{\infty} R_{XX}(\tau)\exp(-j2\pi f\tau)dt$$

- Given the PSD function, the autocorrelation function can be obtained through the inverse Fourier transform

$$R_{XX}(\tau) = F^{-1}\{S_{XX}(f)\} = \int_{-\infty}^{\infty} S_{XX}(f)\exp(j2\pi f\tau)df$$

Properties of the PSD Function

The PSD function called the **spectrum of Random process X(t),** has the properties

- $S_{XX}(f)$ is real and nonnegative
- the average power in **X(t)** is

$$E\{X^2(t)\} = R_{XX}(0) = \int_{-\infty}^{\infty} S_{XX}(f)df$$

- If **X(t)** is real, then $S_{XX}(f)$ is even
- If **X(t)** has periodic components, then $S_{XX}(f)$ will have impulses

Properties of the PSD Function (cont'd)

- Band-related processes
  - Lowpass processes… its psd is zero for **|f|>B**, **B**-bandwidth
  - Bandpass processes… its psd is zero for **f_c-B/2<|f|< f_c+B/2**, **B**-bandwidth, **f_c** – center frequency
- Power and bandwidth calculations
  - The power within an interval

$$P_X(f_1, f_2) = 2\int_{f_1}^{f_2} S_{XX}(f)df$$

  - Effective bandwidth **B_{eff}**, coefficient time **τ_c**

$$B_{eff} = \frac{1}{2}\frac{\int_{-\infty}^{\infty} S_{XX}(f)df}{\max[S_{XX}(f)]} \qquad \tau_c = \frac{\int_{-\infty}^{\infty} R_{XX}(\tau)d\tau}{R_{XX}(0)}$$

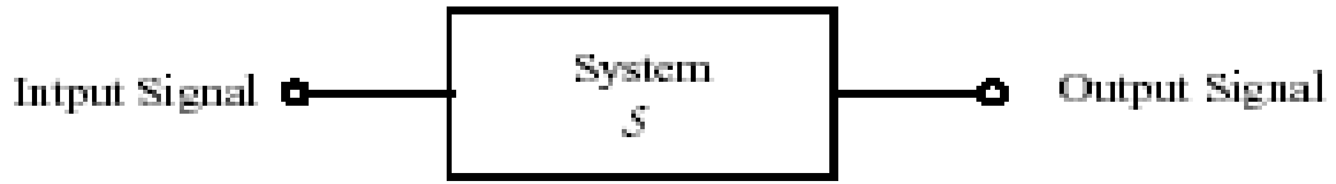
Ergodicity

- **Motivation**
  - Estimation of the ensemble averages from the time averages …
  - Over a single member function of finite duration…
- **Definition**
  - A stationary random process is **ergodic** if its ensemble averages equal (in a mean-square sense) appropriate time averages
  - Ergodicity is related to specific ensemble averages
- **Benefit**
  - Ergodicity means that any ensemble average of **X(t)** can be determined from a single member function of **X(t)** with probability one

MM1. Response of LTI Systems to Random Inputs

Reading page: Chapt 4, pp.216-242

System:

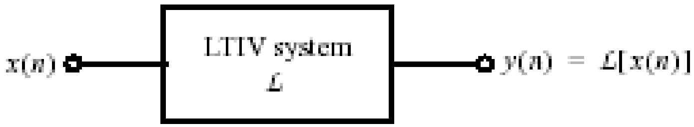


We look at a system as a black box which generates an output signal depending on the input signal and possibly some initial conditions.

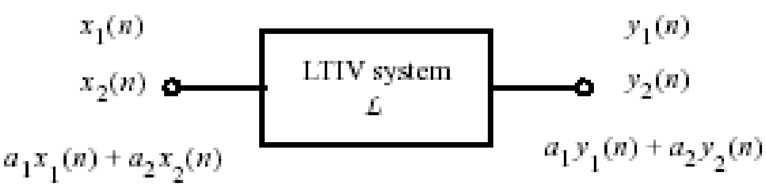
LTI systems

1.1. Discrete-time linear time-invariant (LTIV) systems

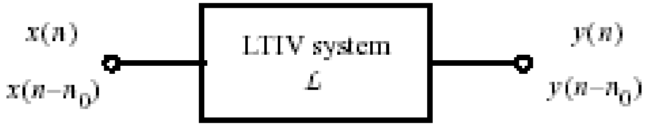
1.1.1. Discrete-time LTIV system



Linear:



Time-invariant:



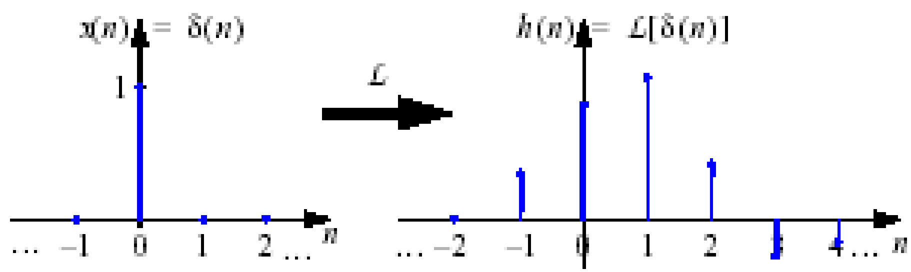
1.1.2. Steady-state description of a LTIV system

- ***Impulse response:***
  The impulse response (IR) $h(n)$ of $\mathcal{L}$ is the response of $\mathcal{L}$ to the unit pulse

$$\delta(n) = \begin{cases} 1\ ; & n = 0 \\ 0\ ; & n \neq 0 \end{cases},$$

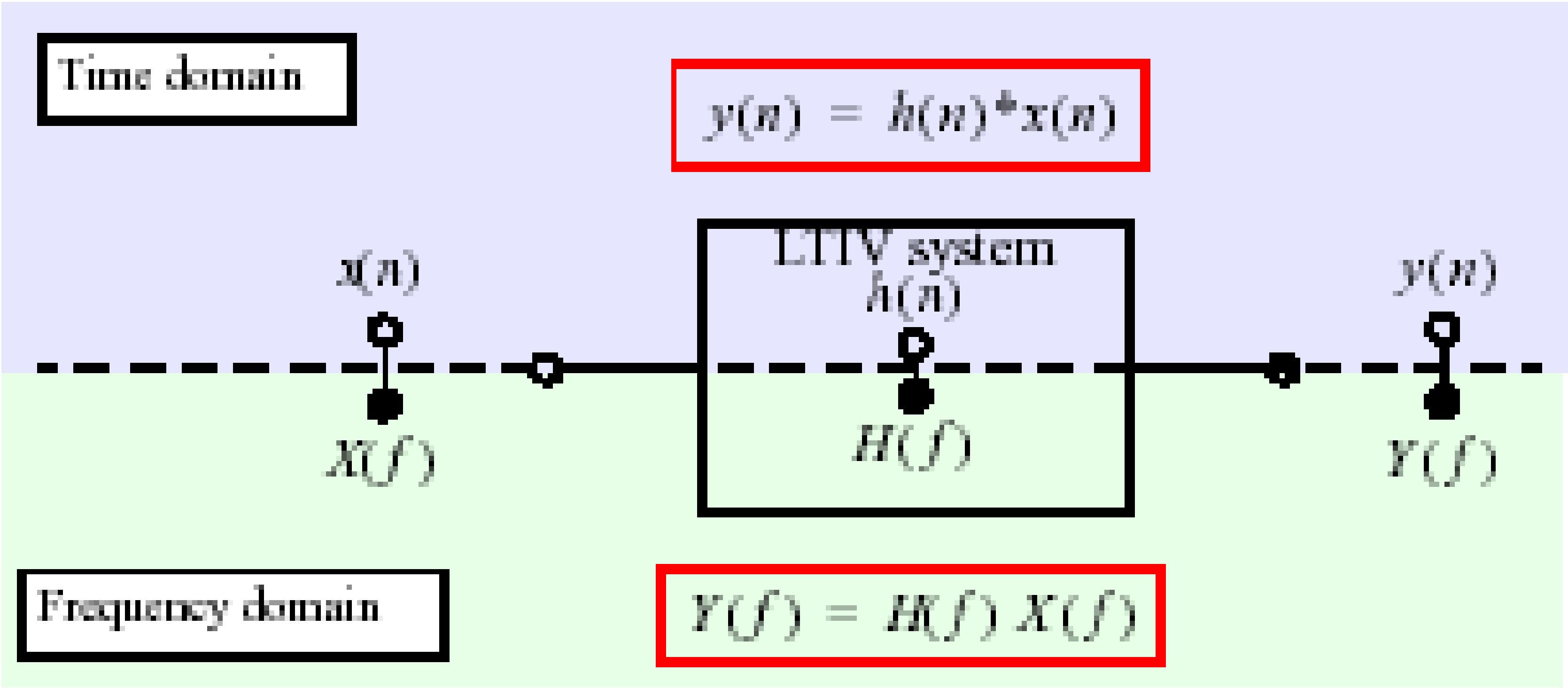
namely

$$h(n) = \mathcal{L}[\delta(n)]$$



LTI systems (Cont'd)

Summary: I-O relationship of a LTIV system:



Response of LTI Systems to Random Inputs

- LTI system: $y[n]=\sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$
- Random inputs
- Two ways for the response computation
  - Compute the response of the LTI system to each member sequence of the random input and then obtain the properties of the ensemble of the output sequences,
  - Computer the properties of the output directly

Response of Discrete-Time Systems

- A causal LTI system with the IR **h[n]/H(f)**

- The output (**random sequence**) **Y[n]** corresponds the random input sequence **X[n]** is

$$Y[n]=X[n]*h[n]=\sum_{k=-\infty}^{\infty} h[k]X[n-k]$$

- The mean sequence of the output **Y[n]** is

$$\mu_Y[n]=E\{Y[n]\}=\sum_{k=-\infty}^{\infty} h[k]E\{X[n-k]\}$$

- The autocorrelation sequence of the output **Y[n]** is

$$R_{YY}[n_1,n_2]= \sum_{k1=-\infty}^{\infty} \sum_{k2=-\infty}^{\infty} h[k_1]h[k_2]\, R_{XX}[n_1-k_1, n_2-k_2]$$

Response of Discrete-Time Systems (Cont'd)

- If **X[n]** is WSS, then the output will also be WSS, i.e.,

$$\mu_Y = E\{Y[n]\} = \Sigma_{k=-\infty}^{\infty} h[k]E\{X[n-k]\} = \mu_X H(0)$$

- The autocorrelation of **Y[n]**:

$$R_{YY}[n_1,n_2] = \Sigma_{k1=-\infty}^{\infty}\Sigma_{k2=-\infty}^{\infty} h[k_1]h[k_2]R_{XX}[(n_2-n_1)-(k_2-k_1)]$$

$$R_{YY}[n] = R_{XX}[n] * h[-n] * h[n] = R_{hh}[n] * R_{XX}[n]$$

- The psd of **Y[n]** is

$$S_{YY}(f) = S_{XX}(f)H(-f)H(f) = S_{XX}(f)|H(f)|^2$$

Response of Discrete-Time Systems (Cont'd)

- The cross-correlation between **X[n]** and **Y[n]** is

$$R_{YX}[n]= \Sigma_{k=-\infty}^{\infty} h[-k]R_{XX}[n-k]=h[-n]*R_{XX}[n]$$

$$R_{XY}[n]= \Sigma_{k=-\infty}^{\infty} h[k]R_{XX}[n-k]=h[n]*R_{XX}[n]$$

$$S_{XY}(f)= H(f)S_{XX}(f)$$

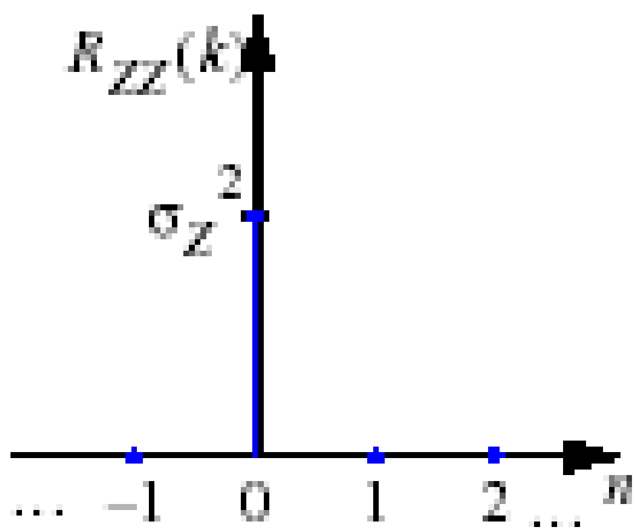
!!!The basis of frequency domain techniques!!!
for the design of LTI systems

Special Case: White noise

White process:

$Z(n)$ is a white process if it satisfies the following conditions:

- $Z(n)$ is a random process
- $\mu_Z(n) = \mathbf{E}[Z(n)] = 0$
- $R_{ZZ}(n, n+k) = \mathbf{E}[Z(n)Z(n+k)] = R_{ZZ}(k) = \sigma_Z^2\delta(k)$

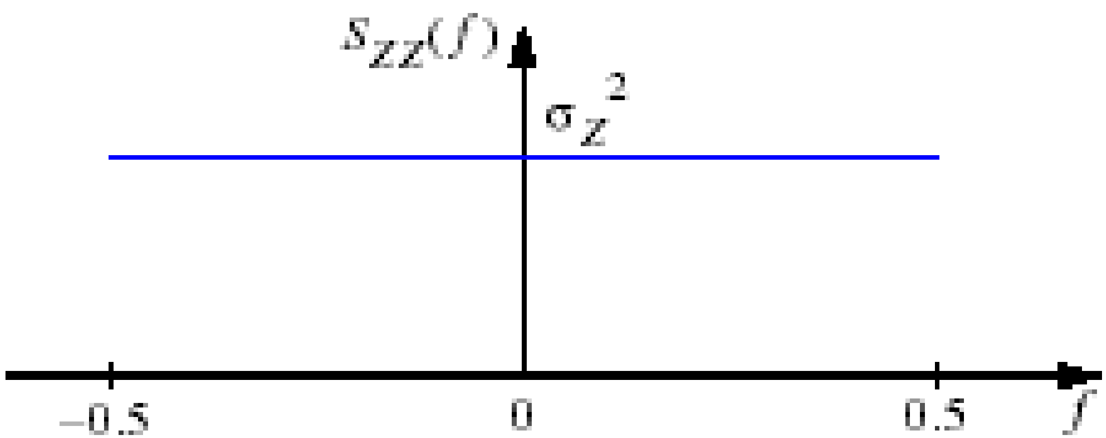


Special Case: White noise (Cont'd)

- ***Spectrum of a white process:***

  If $Z(n)$ is a white process:

$$S_{ZZ}(f) = \sigma_Z^2$$



Estimation of System Parameters

- If the input **X[n]** is a white noise, i.e.,

  $\mu_Y = 0;\ R_{XX}[n] = \sigma^2\,\delta(n)$, or $S_{XX}(f) = \sigma^2$,

According to

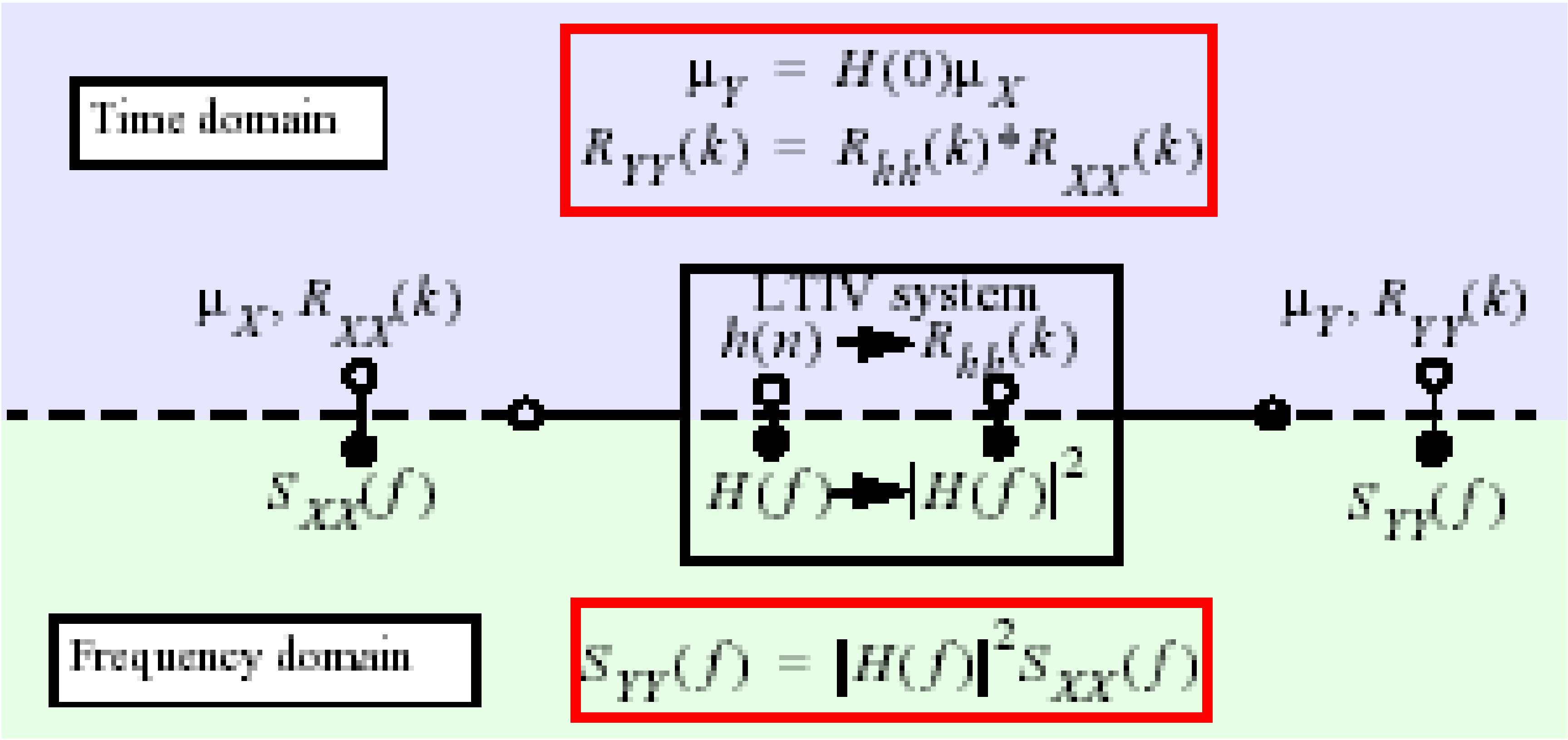
$$R_{XY}[n] = \Sigma_{k=-\infty}^{\infty} h[k] R_{XX}[n-k] = h[n] * R_{XX}[n]$$

There is

$$R_{XY}[n] = \sigma^2\, h[n], \quad \text{or} \quad S_{XY}(f) = \sigma^2\, H(f)$$

- It serves as the basis for estimating the impulse/ frequency response of LTI systems if the output in response to a white-noise input can be observable

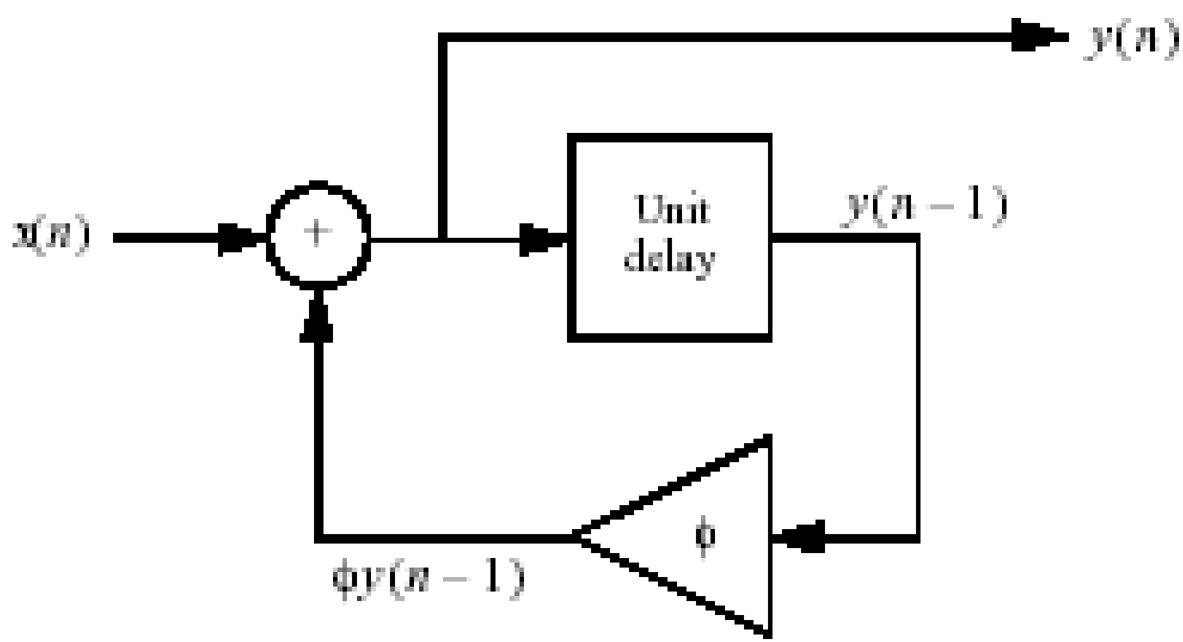
Summary: Second-order I-O relationship of a LTIV system:



Example

1.1.5. Example: First order recursive filter

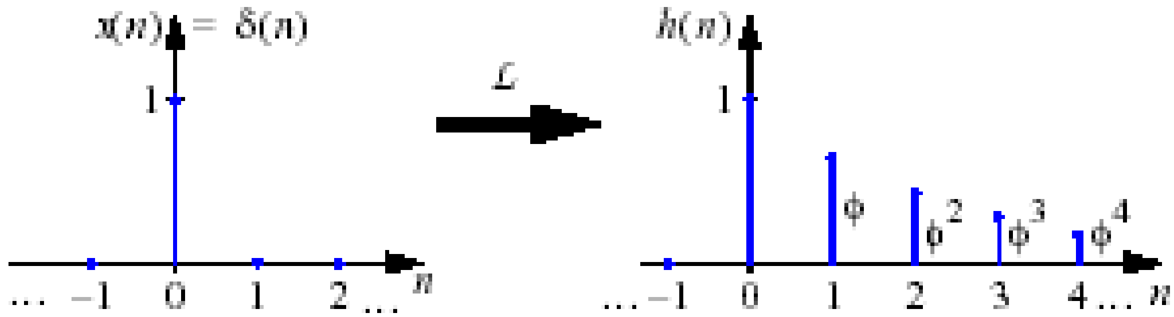
- ***Block diagram and recursive equation:***



$$y(n) = x(n) + \phi y(n-1) \qquad (y(n) = 0 \qquad n < 0)$$

Example (Cont'd)

- ***Impulse response:***



$$h(n) = \begin{cases} 0 \;; & n < 0 \\ \phi^n \;; & n \geq 0 \end{cases}$$

- ***Stability condition:***

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |\phi|^n = \lim_{N \to \infty} \frac{1 - |\phi|^N}{1 - |\phi|} < \infty \quad \Leftrightarrow \quad |\phi| < 1$$

$$\left[\sum_{n=0}^{N} a^n = \frac{1 - a^{N+1}}{1 - a} \right]$$

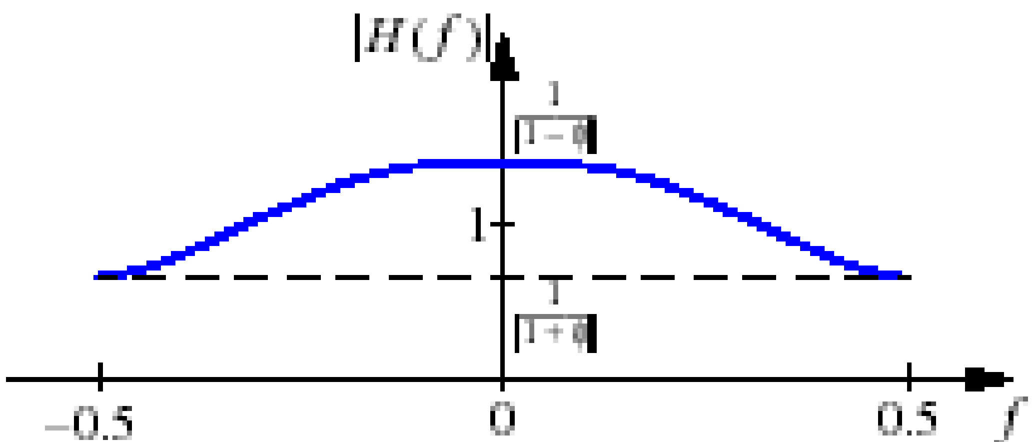
Example (Cont'd)

- ***Transfer function:***

$$H(f) = \mathcal{F}\{h(n)\} = \sum_{n=0}^{\infty} \phi^n \exp(-j2\pi n f)$$

$$= \sum_{n=0}^{\infty} [\phi \exp(-j2\pi f)]^n$$

$$= \frac{1}{1-\phi\exp(-j2\pi f)}$$



Example (Cont'd)

- ***Second-order I-O relationship:***
  - Time domain:

$$\begin{aligned}\mu_Y &= H(0)\mu_X \\ &= \frac{1}{1-\phi}\mu_X\end{aligned}$$

$$\begin{aligned}R_{YY}(k) &= R_{hh}(k) * R_{XX}(k) \\ &= \frac{\phi^{|k|}}{1-\phi^2} * R_{XX}(k)\end{aligned}$$

$$\left[R_{hh}(k) = \sum_{m=0}^{\infty} \phi^m \phi^{m+|k|} = \phi^{|k|} \sum_{m=0}^{\infty} \phi^{2m} = \phi^{|k|} \frac{1}{1-\phi^2}\right]$$

  - Frequency domain:

$$\begin{aligned}S_{YY}(f) &= |H(f)|^2 S_{XX}(f) \\ &= \frac{1}{|1-\phi\exp(-j2\pi f)|^2} S_{XX}(f)\end{aligned}$$

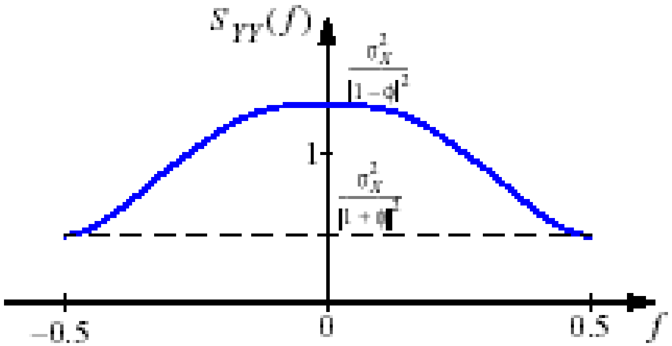
Example (Cont'd)

- ***Special case: AR(1) process (see Section 2.2):***

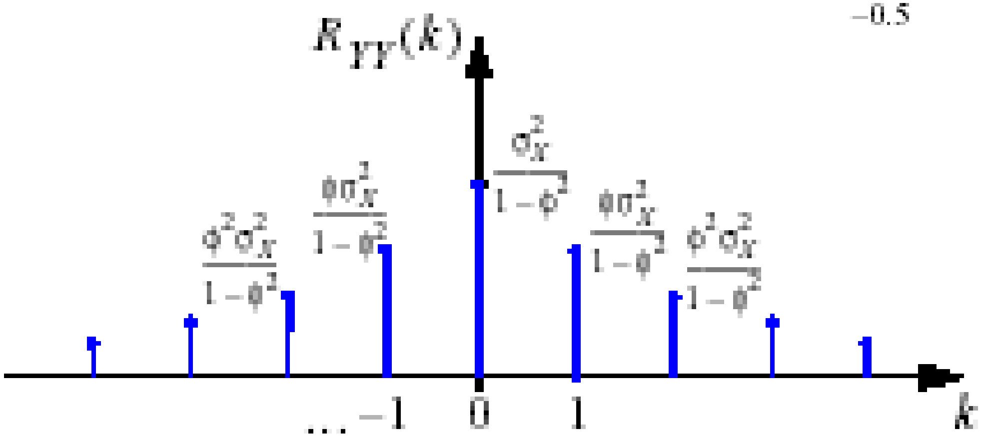
If $X(n)$ is a white Gaussian process,

$$S_{YY}(f) = \frac{\sigma_X^2}{|1 - \phi\exp(-j2\pi f)|^2}$$

$$\mu_Y = 0$$



$$R_{YY}(k) = \frac{\phi^{|k|}}{1 - \phi^2}\sigma_X^2$$



Response of Continuous-Time Systems

- Deterministic case
  - Output of a causal LTI system is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

  - Frequency expression $Y(f) = X(f)H(f)$

- The output (**random process**) **Y(t)** of a causal LTI system in response to a random input process **X(t)** is

$$Y(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$$

Response of Continuous-Time Systems (Cont'd)

- The mean function of the output **Y(t)** is

$$\mu_Y(t) = E\{Y(t)\} = \int_{-\infty}^{\infty} \mu_X(\tau) h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\mu_X(t-\tau)\, d\tau$$

- The autocorrelation function of the output **Y(t)** is

$$R_{YY}(t_1,t_2) = E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(t_1-\tau_1,t_2-\tau_2)\, d\tau_1 d\tau_2$$

Response of Continuous-Time Systems (Cont'd)

- The cross-correlation between **X(t)** and **Y(t)** is

  $$R_{YX}(\tau) = h(-\tau) * R_{XX}(\tau), \text{ and } \quad R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$$

- The autocorrelation of **Y(t)** is

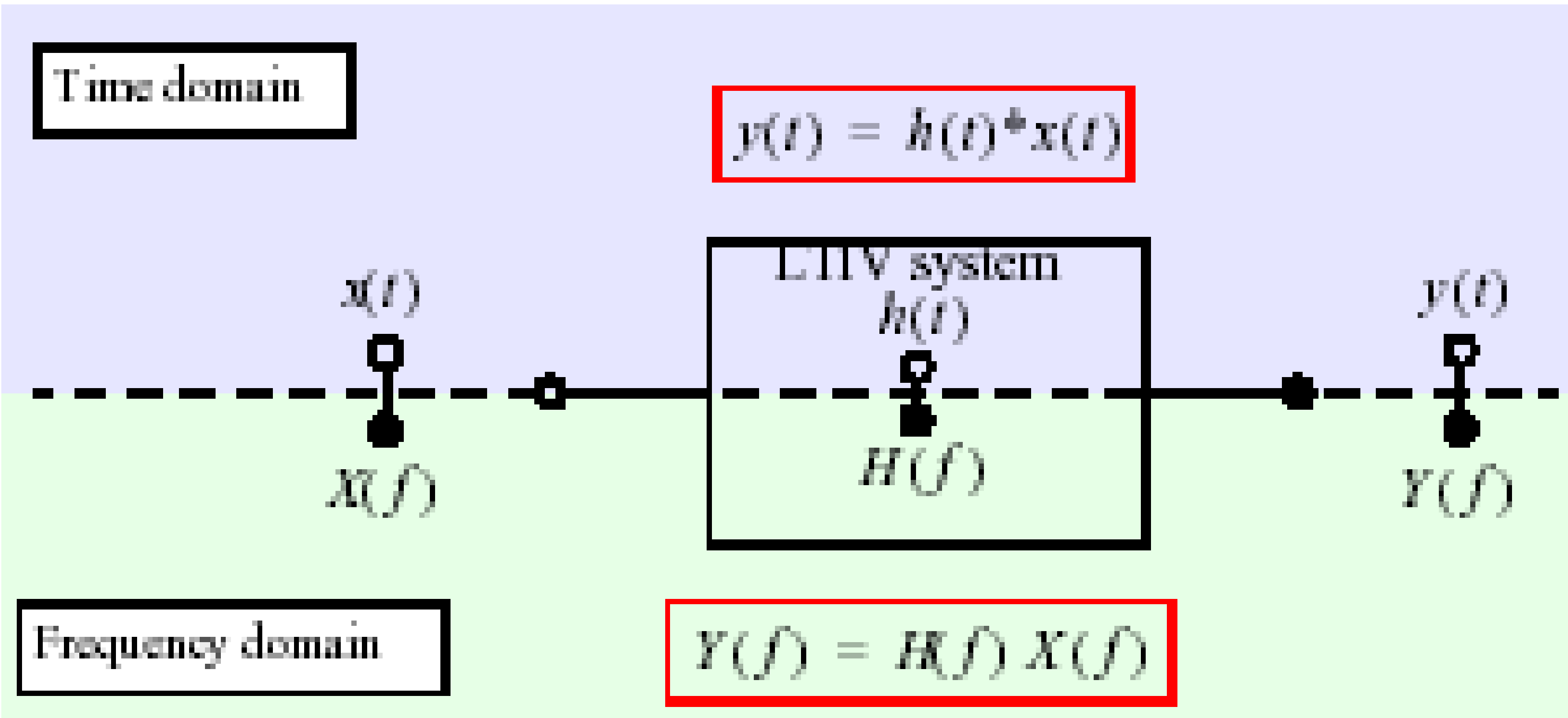
  $$R_{YY}(\tau) = R_{YX}(\tau) * h(\tau) = R_{XX}(\tau) * h(-\tau) * h(\tau)$$

- The psd function of **Y[n]** is

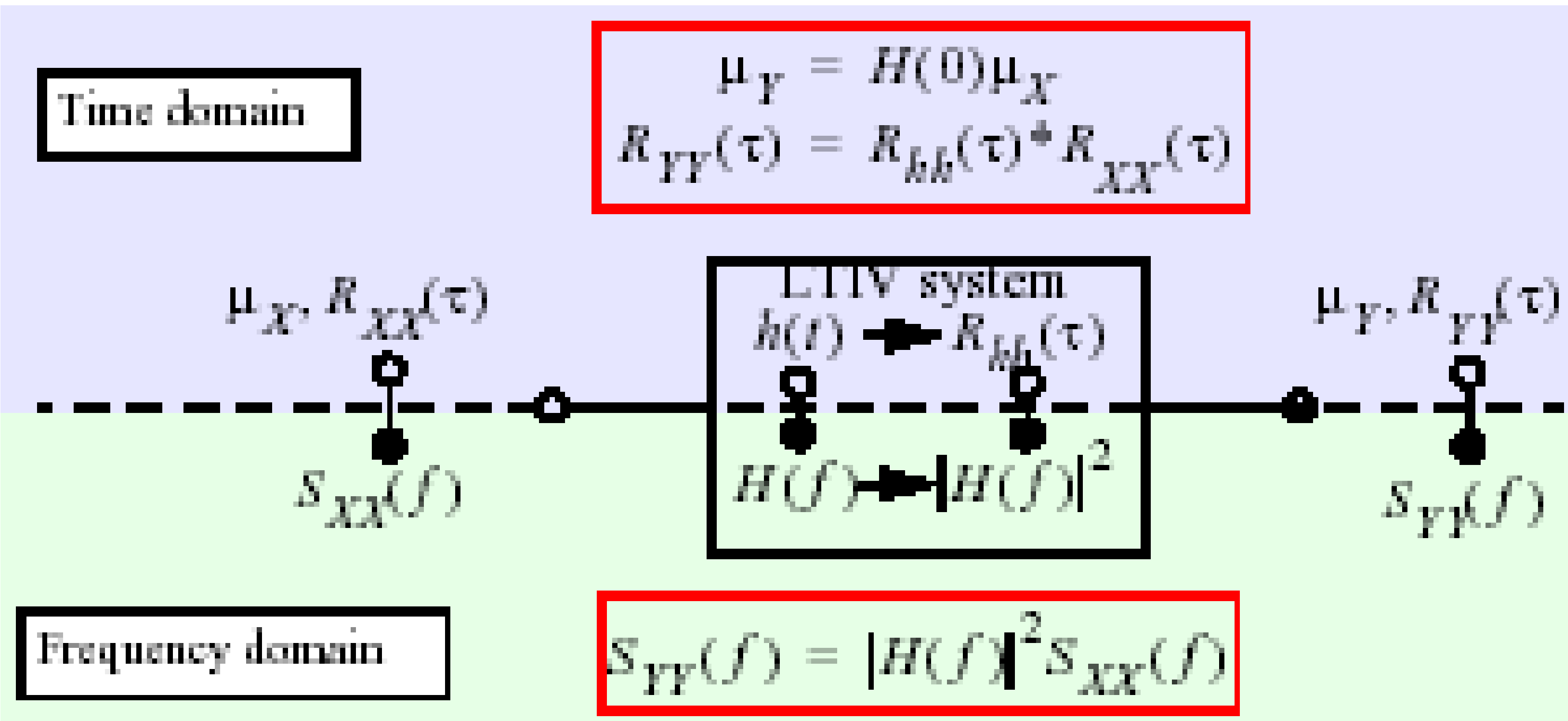
  $$S_{YY}(f) = S_{XX}(f)H(-f)H(f) = S_{XX}(f)|H(f)|^2$$

$|H(f)|^2$ is referred to as the power transfer function

- ***Summary: I-O relation ship of a LTIV system:***



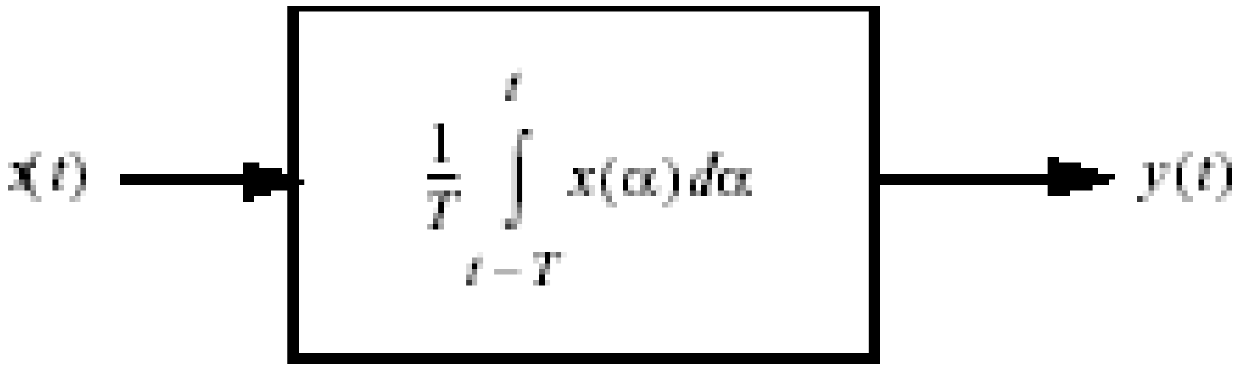
- ***Summary: Second order I-O relationship of a LTIV system:***



Example

1.2.5. Example: Ideal integrator

- ***Block diagram and input-output relationship:***



$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\alpha)\,d\alpha$$