MM2. Discrete Linear Process Models

- Moving Average (MA) models
- Autoregressive (AR) models
- Autoregressive Moving Average (ARMA) models

Reading page: Chapt 5, pp.250-275

Motivation

- Prototype signals: impulse, exponential signals
- Signals generated by prototype signals through a system
- System identification
- Data Analysis

Discrete Linear Process Models

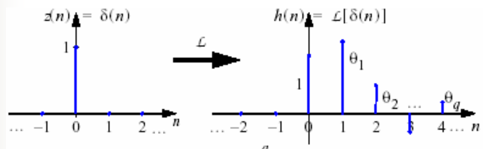
- **Moving Average (MA) models**
- Autoregressive (AR) models
- Autoregressive Moving Average (ARMA) models

Moving Average (MA) Processes

- Definition: A random sequence X[n] is a moving average process of order q (MA(q)) if for any n, there is

$$X[n] = Z[n] + \sum_{i=1}^{q} \theta_i Z[n-i] \qquad \theta_q \neq 0$$

Where Z[n] is a white Gaussian process



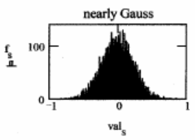
$$h(n) = \delta(n) + \sum_{i=1}^{q} \theta_i \delta(n-i)$$

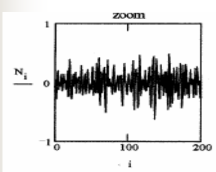
White Gaussian Noise

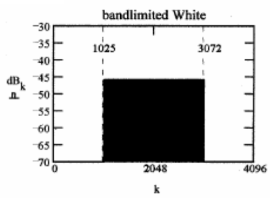
$$E\{Z(n)\} = 0$$

$$E\{Z(i)Z(j)\} = \begin{cases} \sigma_Z^2 & for \quad i = j \\ 0 & for \quad i \neq j \end{cases}$$

$$f_{Z(n)}(\lambda) = \frac{1}{\sqrt{2\pi}\sigma_Z} \exp\left(-\frac{\lambda^2}{\sigma_Z^2}\right)$$





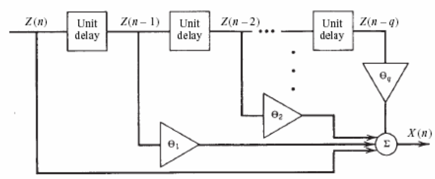


Properties of MA Models

- Stability
- Transfer function:

$$H(z) = 1 + \sum_{i=1}^{q} \theta_i z^{-i} \qquad H(f) = 1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f)$$

- *Transversal filter implementation of a MA(q) process:*



Statistic Properties of MA Process

- Mean, autocorrelation and PSD functions of MA(q) process:

$$\mu_X = 0$$

$$R_{XX}[k] = \sigma_Z{}^2 R_{hh}[k] = \begin{cases} \sigma_Z^2[\theta_k + \sum_{j=k+1}^{q} \theta_j \theta_{j-k}] & k < q \\ \sigma_Z^2 \theta_q & k = q \\ 0 & k > q \end{cases}$$

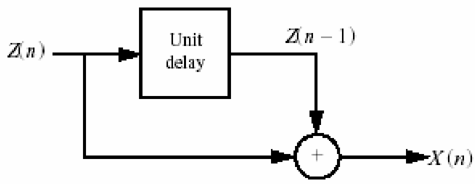
$$where \quad R_{hh}[k] = h[k] * h[-k]$$

$$S_{XX}(f) = |1 + \sum_{i=1}^{q} \theta_i \exp(-j2\pi i f)|^2 \sigma_Z{}^2$$

See p.270

Example: MA process

Example: MA(1)

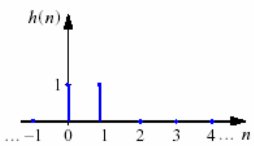


$$X(n) = Z(n) + Z(n-1) \qquad (\theta_1 = 1)$$

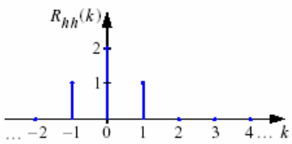
Example: MA process (cont'd)

- Impulse response and autocorrelation function of the transversal filter

$$h(n) = \begin{cases} 1\ ; & n \in \{0, 1\} \\ 0\ ; & \text{elsewhere} \end{cases}$$



$$R_{hh}(k) = \begin{cases} 2\ ; & k \in \{0\} \\ 1\ ; & k \in \{-1, 1\} \\ 0\ ; & \text{elsewhere} \end{cases}$$

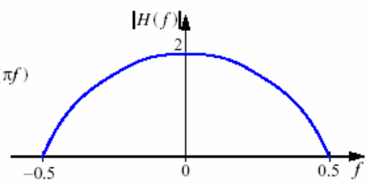


Example: MA process (cont'd)

- Transfer function:

$$H(f) = 1 + \exp(-j2\pi f) \qquad |f| \le 0.5$$
$$= \exp(-j\pi f)[\exp(j\pi f) + \exp(-j\pi f)]$$
$$= 2\exp(-j\pi f)\cos(\pi f)$$

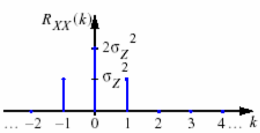
$$|H(f)| = 2\cos(\pi f)$$
$$(|f| \le 0.5)$$



Example: MA process (cont'd)

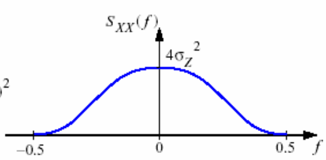
- Autocorrelation function of $X(n)$:

$$R_{XX}(k) = \sigma_Z{}^2 R_{hh}(k)$$
$$= \begin{cases} 2\sigma_Z{}^2\ ; & k \in \{0\} \\ \sigma_Z{}^2\ ; & k \in \{-1, 1\} \\ 0\ ; & \text{elsewhere} \end{cases}$$



- Power spectrum of $X(n)$:

$$S_{XX}(f) = \sigma_Z{}^2 |H(f)|^2$$
$$= 4\sigma_Z{}^2 \cos(\pi f)^2$$



Discrete Linear Process Models

- Moving Average (MA) models
- **Autoregressive (AR) models**
- Autoregressive Moving Average (ARMA) models

Autoregressive (AR) Models

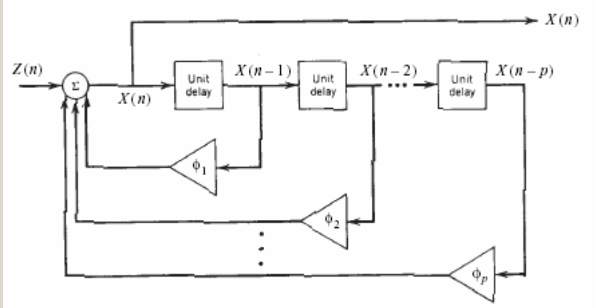
- Definition: A random sequence X[n] is an autoregressive process of order p (AR(p)) if it is WSS and for any n, there is

$$X(n) = \sum_{i=1}^{p} \varphi_i X(n-i) + Z(n)$$

Where Z[n] is a white Gaussian process

- Recursive filters, all-pole models, state space model, ..

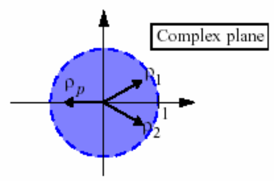
Recursive Filter Implementation



Poles of AR processes (all-pole model)

- **Stability:** the AR process X[n] is causal and stable, if and only if, the poles of the following transfer function locate inside the unit circle.

$$H(z) = \frac{1}{1 - \sum_{i=1}^{p} \varphi_i z^{-i}} \qquad H(f) = \frac{1}{1 - \sum_{i=1}^{p} \varphi_i \exp(-j2\pi i f)}$$



State Space Model

$$\mathbf{X}(n) = \begin{bmatrix} X(n) \\ X(n-1) \\ \vdots \\ X(n-p+1) \end{bmatrix} \qquad \mathbf{Z}(n) = \begin{bmatrix} Z(n) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\mathbf{X}(n) = \mathbf{A}\mathbf{X}(n-1) + \mathbf{Z}(n)$$

where

$$A = \begin{bmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_p \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots 1 & 0 \end{bmatrix}$$

Statistic Properties of AR processes

- **Mean, autocorrelation and PSD functions of** AR process X[n]:

$$\mu_X = 0$$

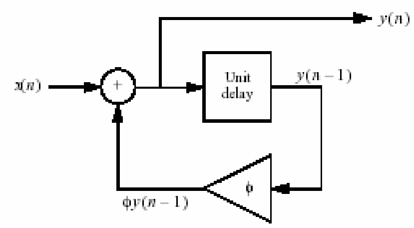
$$R_{XX}[k] = \sigma_Z^{\ 2} R_{hh}[k]$$

$$S_{XX}(f) = \frac{\sigma_Z^{\ 2}}{|1 - \sum_{i=1}^{p} \varphi_i \exp(-j2\pi i f)|^2}$$

Example – AR process

1.1.5. Example: First order recursive filter

- ***Block diagram and recursive equation:***



See the blackboard

$y(n) = x(n) + \phi y(n-1) \qquad (y(n) = 0 \quad n < 0)$

Yule-Walker Equation

$k \geq 0$:

$$X(n) = \sum_{i=1}^{p} \phi_i X(n-i) + Z(n)$$

$$X(n)\,X(n-k) = \sum_{i=1}^{p} \phi_i X(n-i)X(n-k) + Z(n)X(n-k)$$

$$\mathbf{E}[X(n)\,X(n-k)] = \sum_{i=1}^{p} \phi_i \mathbf{E}[X(n-i)X(n-k)] + \mathbf{E}[Z(n)X(n-k)$$

$$R_{XX}(n,n-k) = \sum_{i=1}^{p} \phi_i R_{XX}(n-i,n-k) + \sigma_Z^2 \delta(k)$$

$$R_{XX}(k) = R_{XX}(-k) = \sum_{i=1}^{p} \phi_i R_{XX}(i-k) + \sigma_Z^2 \delta(k)$$

Using a vector notation, for $0 \leq k \leq p$

$$R_{XX}(k) = [R_{XX}(1-k), \ldots, R_{XX}(p-k)] \begin{bmatrix} \phi_1 \\ \ldots \\ \phi_p \end{bmatrix} + \sigma_Z^2 \delta(k) \quad (2.1)$$

For $k > p$:

$$R_{XX}(k) = [R_{XX}(k-1), \ldots, R_{XX}(k-p)] \begin{bmatrix} \phi_1 \\ \ldots \\ \phi_p \end{bmatrix} \quad (2.2)$$

Let us define

$$\Phi = \begin{bmatrix} \phi_1 \\ \ldots \\ \phi_p \end{bmatrix} \qquad \gamma = \begin{bmatrix} R_{XX}(1) \\ R_{XX}(2) \\ \ldots \\ R_{XX}(p) \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \ldots & R_{XX}(p-1) \\ R_{XX}(-1) & R_{XX}(0) & \ldots & R_{XX}(p-2) \\ \ldots & \ldots & \ldots & \ldots \\ R_{XX}(-(p-1)) & R_{XX}(-(p-2)) & \ldots & R_{XX}(0) \end{bmatrix}$$

Note that Γ is symmetric.

Then, for $k = 0$ Identity (2.1) becomes

$$R_{XX}(0) = \gamma^T \Phi + \sigma_Z^2$$

Inserting $k = 1, \ldots, p$ in (2.1) yields p identities that can be concatenated in a matrix form according to

$$\begin{bmatrix} R_{XX}(1) \\ R_{XX}(2) \\ \ldots \\ R_{XX}(p) \end{bmatrix} = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \ldots & R_{XX}(p-1) \\ R_{XX}(-1) & R_{XX}(0) & \ldots & R_{XX}(p-2) \\ \ldots & \ldots & \ldots & \ldots \\ R_{XX}(-(p-1)) & R_{XX}(-(p-2)) & \ldots & R_{XX}(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \ldots \\ \phi_p \end{bmatrix}$$

$$\gamma = \Gamma\Phi$$

System identification
Yule walker (autocorre.)

Comments:

- The feed-back coefficients $\phi_1, \ldots, \phi_p$ of the recursiv filter and the variance σ_Z^2 of the white Gaussian input process $Z(n)$ can be computed from $R_{XX}(0), \ldots, R_{XX}(p)$ via the Yule-Walker equations and vice-versa.
- The samples $R_{XX}(k)$, $k > p$ can be recursively computed from $\phi_1, \ldots, \phi_p$ and $R_{XX}(k-1), \ldots, R_{XX}(k-p)$ by using Identity (2.2).

Discrete Linear Process Models

- Moving Average (MA) models
- Autoregressive (AR) models
- **Autoregressive Moving Average (ARMA) models**

Autoregressive Moving Average (ARMA) Models

- Definition: A random sequence X[n] is an autoregressive moving average (ARMA) process of order (p,q) (denoted as ARMA(p,q) if it is WSS and for any n, there is

$$X(n) = \sum_{i=1}^{p} \varphi_i X(n-i) + \sum_{j=1}^{q} \theta_j Z(n-j) + Z(n)$$

Where Z[n] is a white Gaussian process

- Any WSS process can be approximated by an ARMA(p,q) process

Properties of ARMA Processes

- Transfer function:

$$H(z) = \frac{1 + \sum_{k=1}^{q} \theta_k z^{-k}}{1 - \sum_{i=1}^{p} \varphi_i z^{-i}} \qquad H(f) = \frac{1 + \sum_{k=1}^{q} \theta_k e^{-j2k\pi f}}{1 - \sum_{i=1}^{p} \varphi_i e^{-j2i\pi f}}$$

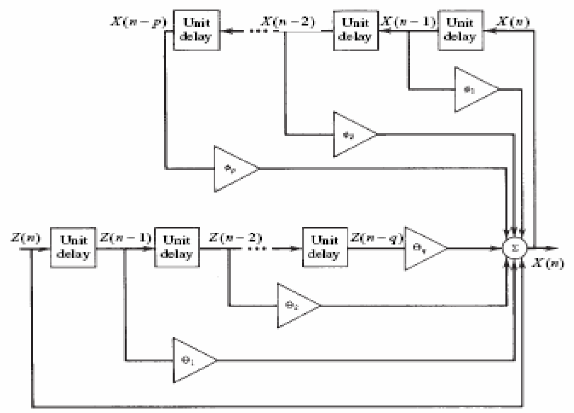
- Stability and causality
- Statistic properties:

$$\mu_X = 0$$

$$R_{XX}[k] = \sigma_Z^2 R_{hh}[k]$$

$$S_{XX}(f) = \frac{|1 + \sum_{k=1}^{q} \varphi_k \exp(-j2\pi k f)|^2}{|1 - \sum_{i=1}^{p} \varphi_i \exp(-j2\pi i f)|^2} \sigma_Z^2$$

- ***Filter implementation:***



Remark: AR,ARX and ARMAX Models

- System Identification Toolbox:
  - Estimate the parameters of an AR model for scalar time series:

    m = ar(y,n)

    $$A(q)y(t) = e(t) \quad \Leftrightarrow \quad A(z^{-1})X(n) = Z(n) \quad A(z^{-1}) = 1 - \sum_{i=1}^{p} \varphi_i z^{-1}$$

  - Estimate the parameters of an ARX or AR model:

    m = arx(data,'na',na,'nb',nb,'nk',nk)

    $$A(q)y(t) = B(q)u(t-n_k) + e(t) \quad \Leftrightarrow \quad A(z^{-1})X(n) = B(z^{-1})U(n-n_k) + Z(n)$$

  - Estimate the parameters of an ARMAX or ARMA model:

    m = armax(data,'na',na,'nb',nb,'nc',nc,'nk',nk)

    $$A(q)y(t) = B(q)u(t-n_k) + C(q)e(t) \quad \Leftrightarrow \quad A(z^{-1})X(n) = B(z^{-1})U(n-n_k) + C(z^{-1})Z(n)$$

Summary of MM2

- MA(q): $X[n] = Z[n] + \sum_{i=1}^{q} \theta_i Z[n-i] \qquad \theta_q \neq 0$
- AR(p): $X(n) = \sum_{i=1}^{p} \varphi_i X(n-i) + Z(n)$
- ARMA(p,q): $X(n) = \sum_{i=1}^{p} \varphi_i X(n-i) + \sum_{j=1}^{q} \theta_j Z(n-j) + Z(n)$
- Statistic properties

  $$\mu_X = 0$$

  $$R_{XX}[k] = \sigma_Z{}^2 R_{hh}[k]$$

  $$S_{XX}(f) = \frac{|1 + \sum_{k=1}^{q} \varphi_k \exp(-j2\pi kf)|^2}{|1 - \sum_{i=1}^{p} \varphi_i \exp(-j2\pi if)|^2} \sigma_Z{}^2$$