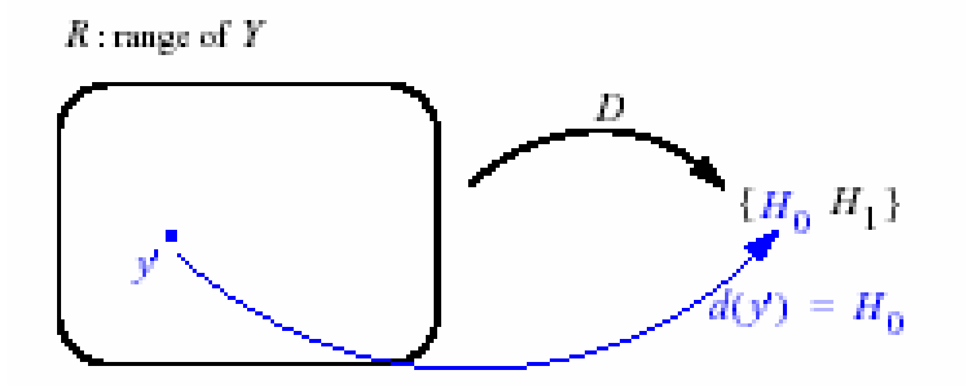
MM4. Signal Detection (Part Two)

Reading page: Chapt 6, pp.352-361, 366-370

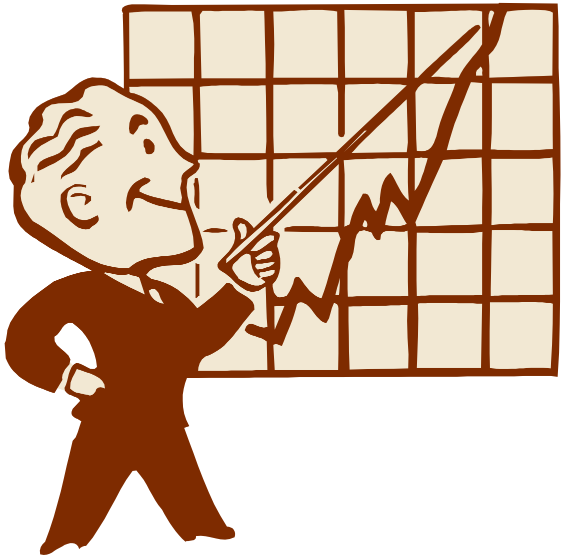
- Explain MM3 exercise
- 4.1 Binary detection of discrete-time signals
- 4.2 Binary detection of continuous-time signals
- 4.3 M-ary detection
- Remark of self-study presentation

What have we talked through MM3?



3.1 Hypothesis Testing

In hypothesis testing, a **decision** is made based on the **observation** of a random variable as to which of several **hypotheses** to accept



3.2 Decision Rules

- Maximum ”a posteriori” decision rule
- Bayes decision rule – Costs of errors
- Minmax rule and Neyman-Pearson rule

Maximum "a posteriori" (MAP) Rule

- MAP decision rule:

$$f(y|H_1)P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} f(y|H_0)P(H_0)$$

- Prior distribution $P(H_i)$, i=1,2

- Likelihood ratio L(y):

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

$$l(y) = \ln(L(y))$$

$$l(y) = \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

Bayes' Decision Rule – Cost of Errors

- Motivation: cost of wrong decisions

Cost function:
In many engineering branches costs have to be taken into account depending on the decision and the true hypothesis.

Decision D	True hypothesis	
	H_0	H_1
H_0	C_{00}	C_{01}
H_1	C_{10}	C_{11}

Usually, the cost of making a wrong decision is higher than that of making a correct decision:

$$C_{10} \geq C_{00} \qquad \text{and} \qquad C_{01} \geq C_{11}.$$

Bayes' Decision Rule

- Average cost:

$$\bar{C} = C_{00}P[D = H_0 | H_0]P[H_0] + C_{10}P[D = H_1 | H_0]P[H_0] + $$
$$+ C_{01}P[D = H_0 | H_1]P[H_1] + C_{11}P[D = H_1 | H_1]P[H_1]$$

- Bayes' decision rule: minimize the average cost

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)(C_{10} - C_{00})}{P(H_1)(C_{01} - C_{11})}$$

Explain MM3 Exercise!

Q-function for Gaussian Distribution

- Gaussian pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X{}^2}} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X{}^2}\right\}$$

- Probability distribution:

$$P(X > a) = \int_a^\infty f_X(x)dx = \int_a^\infty \frac{1}{\sqrt{2\pi\sigma_X{}^2}} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X{}^2}\right\}dx$$

$$P(X > a) = \int_{(a-\mu_X)/\sigma_X}^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}dx$$

- Q-function:

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \exp\left\{-\frac{z^2}{2}\right\}dz$$

4.1 Binary detection of discrete-time signals

-**Binary detection with multiple observations**

-***Time-limited case***

-*Time-infinite but finite energy case*

Time-limited: **Assumptions**

- Singal model: Y(n)=x(n)+W(n), n=0,1,…N-1

$$x(n) = \begin{cases} s_0(n) & \textit{under hypothesis } H_0 \\ s_1(n) & \textit{under hypothesis } H_1 \end{cases}$$

- W(n) is a white Guaussian noise:

$$W(n) \quad \textit{is a Gaussian process}$$
$$E\{W(n)\} = 0$$
$$R_{WW}(k) = \sigma^2 \delta(k)$$

Time-limited: **Notations**

- Vector expressions:

$$\mathbf{Y} = \begin{bmatrix} Y_0 & Y_1 & \cdots & Y_{N-1} \end{bmatrix}^T \quad \mathbf{y} = \begin{bmatrix} y_0 & y_1 & \cdots & y_{N-1} \end{bmatrix}^T$$

$$\mathbf{S}_0 = \begin{bmatrix} s_{00} & s_{01} & \cdots & s_{0N-1} \end{bmatrix}^T \quad \mathbf{S}_1 = \begin{bmatrix} s_{10} & s_{11} & \cdots & s_{1N-1} \end{bmatrix}^T$$

$$\mathbf{W} = \begin{bmatrix} W_0 & W_1 & \cdots & W_{N-1} \end{bmatrix}^T$$

- Considered signal in vector form:

$$\mathbf{Y} = \mathbf{X} + \mathbf{W}$$

Time-limited: **PDF functions**

- Pdf function of **W**:

$$f(\mathbf{w}) = \prod_{n=0}^{N-1} f(w_n) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi\sigma}} \exp\{-\frac{w_n^{\ 2}}{2\sigma^2}\}$$

$$f(\mathbf{w}) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^N \exp\left\{-\frac{\|\mathbf{w}\|^2}{2\sigma^2}\right\} \quad \|\mathbf{w}\|^2 = \sum_{n=0}^{N-1} w_n^{\ 2}$$

- Conditional pdf functions:

$$f(\mathbf{y}\,|\,H_i) = f(\mathbf{w})\,|_{\mathbf{w}=\mathbf{y}-\mathbf{s_i}}$$

$$f(\mathbf{y}\,|\,H_i) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^N \exp\left\{-\frac{\|\mathbf{y}-\mathbf{s_i}\|^2}{2\sigma^2}\right\} \quad \|\mathbf{y}-\mathbf{s_i}\|^2 = \sum_{n=0}^{N-1} (y_n - s_{in})^2 \quad i = 1,2$$

Time-limited: Likelihood Ratios

- Likelihood ratio:

$$L(\mathbf{y}) = \frac{f(\mathbf{y}|H_1)}{f(\mathbf{y}|H_0)} = \frac{\exp\left\{-\frac{\|\mathbf{y}-\mathbf{s_1}\|^2}{2\sigma^2}\right\}}{\exp\left\{-\frac{\|\mathbf{y}-\mathbf{s}_0\|^2}{2\sigma^2}\right\}} = \exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right]\right\}$$

- Loglikelihood ratio:

$$l(\mathbf{y}) = \ln\left(\frac{f(\mathbf{y}|H_1)}{f(\mathbf{y}|H_0)}\right) = \ln(\exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right]\right\})$$

$$l(\mathbf{y}) = \frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right] = \frac{1}{\sigma^2}\left[\mathbf{y}^{\mathbf{T}}(\mathbf{s_1}-\mathbf{s_0}) + \frac{1}{2}(\|\mathbf{s}_0\|^2 - \|\mathbf{s}_1\|^2)\right]$$

Time-limited: Decision Rules

- Decision rule:

$$l(\mathbf{y}) = \frac{1}{\sigma^2}\left[\mathbf{y}^T(\mathbf{s}_1 - \mathbf{s}_0) + \frac{1}{2}\left(\|\mathbf{s}_0\|^2 - \|\mathbf{s}_1\|^2\right)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\gamma)$$

$$\mathbf{y}^T(\mathbf{s}_1 - \mathbf{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}^{\;2}$$

$$\sum_{n=0}^{N-1} y_n (s_{1n} - s_{0n}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right)$$

- MAP decision rule:

$$\mathbf{y}^T(\mathbf{s}_1 - \mathbf{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{P(H_0)}{P(H_1)}\right) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}^{\;2}$$

4.1 Binary detection of discrete-time signals

-**Binary detection with multiple observations**

-***Time-limited case***

-***Time-infinite but finite energy case***

Finite-energy: Assumptions

- Singal model: Y(n)=x(n)+W(n), n=…,-2.-1.0,1,…N-1,….

$$x(n) = \begin{cases} s_0(n) & \textit{under hypothesis } H_0 \\ s_1(n) & \textit{under hypothesis } H_1 \end{cases}$$

$$E_{s_i} = \sum_{n=-\infty}^{\infty} s_{in}^{\ 2} = \| s_{in} \|^2 < \infty$$

- W(n) is a white Guaussian noise:

$$W(n) \quad \textit{is a Gaussian process}$$

$$E\{W(n)\} = 0$$

$$R_{WW}(k) = \sigma^2 \delta(k)$$

Finite-energy: Notations

- Vector expressions:

$$\mathbf{Y} = \begin{bmatrix} \cdots & Y_0 & Y_1 & \cdots & Y_{N-1} & \cdots \end{bmatrix}^T \quad \mathbf{y} = \begin{bmatrix} \cdots & y_0 & y_1 & \cdots & y_{N-1} & \cdots \end{bmatrix}^T$$

$$\mathbf{S}_0 = \begin{bmatrix} \cdots & s_{00} & s_{01} & \cdots & s_{0N-1} & \cdots \end{bmatrix}^T \quad \mathbf{S}_1 = \begin{bmatrix} \cdots & s_{10} & s_{11} & \cdots & s_{1N-1} & \cdots \end{bmatrix}^T$$

$$\mathbf{W} = \begin{bmatrix} \cdots & W_0 & W_1 & \cdots & W_{N-1} & \cdots \end{bmatrix}^T$$

- Considered signal in vector form:

$$\mathbf{Y} = \mathbf{X} + \mathbf{W}$$

Finite-energy: **PDF functions**

- Pdf function of **W:**

$$f(\mathbf{w}) = \prod_{n=-\infty}^{\infty} f(w_n) = \prod_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} \exp\{-\frac{w_n^{\ 2}}{2\sigma^2}\}$$

- Conditional pdf functions:

$$f(\mathbf{y} \mid H_i) = f(\mathbf{w})\,|_{\mathbf{w}=\mathbf{y}-\mathbf{s_i}}$$

$$f(\mathbf{y} \mid H_i) = \left(\frac{1}{\sqrt{2\pi\sigma}}\right)^{\infty} \exp\left\{-\frac{\|\mathbf{y}-\mathbf{s_i}\|^2}{2\sigma^2}\right\} \quad \|\mathbf{y}-\mathbf{s_i}\|^2 = \sum_{n=-\infty}^{\infty} (y_n - s_{in})^2 \quad i = 1,2$$

Finite-energy: Likelihood Ratios

- Likelihood ratio:

$$L(\mathbf{y}) = \frac{f(\mathbf{y}\,|\,H_1)}{f(\mathbf{y}\,|\,H_0)} = \frac{\exp\left\{-\dfrac{\|\mathbf{y}-\mathbf{s_1}\|^2}{2\sigma^2}\right\}}{\exp\left\{-\dfrac{\|\mathbf{y}-\mathbf{s}_0\|^2}{2\sigma^2}\right\}} = \exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right]\right\}$$

- Loglikelihood ratio:

$$l(\mathbf{y}) = \ln\left(\frac{f(\mathbf{y}\,|\,H_1)}{f(\mathbf{y}\,|\,H_0)}\right) = \ln(\exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right]\right\})$$

$$l(\mathbf{y}) = \frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right] = \frac{1}{\sigma^2}\left[\mathbf{y}^{\mathbf{T}}(\mathbf{s_1}-\mathbf{s_0}) + \frac{1}{2}(\|\mathbf{s}_0\|^2 - \|\mathbf{s}_1\|^2)\right]$$

Finite-energy: Decision Rules

- Decision rule:

$$l(\mathbf{y}) = \frac{1}{\sigma^2}\left[\mathbf{y}^T(\mathbf{s_1}-\mathbf{s_0}) + \frac{1}{2}\left(\|\mathbf{s_0}\|^2 - \|\mathbf{s}_1\|^2\right)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\gamma)$$

$$\mathbf{y}^T(\mathbf{s_1}-\mathbf{s_0}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2\ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}{}^2$$

$$\sum_{n=-\infty}^{\infty} y_n(s_{1n} - s_{0n}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2\ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right)$$

- MAP decision rule:

$$\mathbf{y}^T(\mathbf{s_1}-\mathbf{s_0}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2\ln\left(\frac{P(H_0)}{P(H_1)}\right) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}{}^2$$

4.2 Binary detection of continuous-time signals

-**Binary detection with continuous observations**

-***Bandlimited case***

-***Time-infinite but finite energy case***

Bandlimited: Assumptions

- Singal model: Y(t)=x(t)+W(t),

$$x(t) = \begin{cases} s_0(t) & under\ hypothesis\ H_0 \\ s_1(t) & under\ hypothesis\ H_1 \end{cases}$$

$$E_{s_i} = \int_{-\infty}^{\infty} (s_i(t))^2\, dt < \infty \quad and \quad s_i(t)\ is\ band\ \lim ited \quad i = 1,2$$

- W(t) is a white Guaussian noise:

$$W(t) \quad is\ a\ Gaussian\ process$$

$$E\{W(t)\} = 0$$

$$R_{WW}(\tau) = \frac{N_0}{2}\delta(\tau) \quad S_{WW}(f) = \frac{N_0}{2}$$

Bandlimited: Decision Rules

- Decision rule:

$$\int_{-\infty}^{\infty} y(t)(s_1(t) - s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \int_{-\infty}^{\infty} (s_i(t))^2 dt$$

- MAP decision rule:

$$\int_{-\infty}^{\infty} y(t)(s_1(t) - s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\left(\frac{P(H_0)}{P(H_1)}\right) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \int_{-\infty}^{\infty} (s_i(t))^2 dt$$

Timelimited: **Decision Rules**

- Time-limited but possibly bandwidth unlimited finite-energy signals
- Decision rule:

$$\int_{-T_s}^{T_e} y(t)(s_1(t)-s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln(\gamma)+\frac{1}{2}\left(E_{\mathbf{s}_1}-E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \int_{T_s}^{T_e} (s_i(t))^2\,dt$$

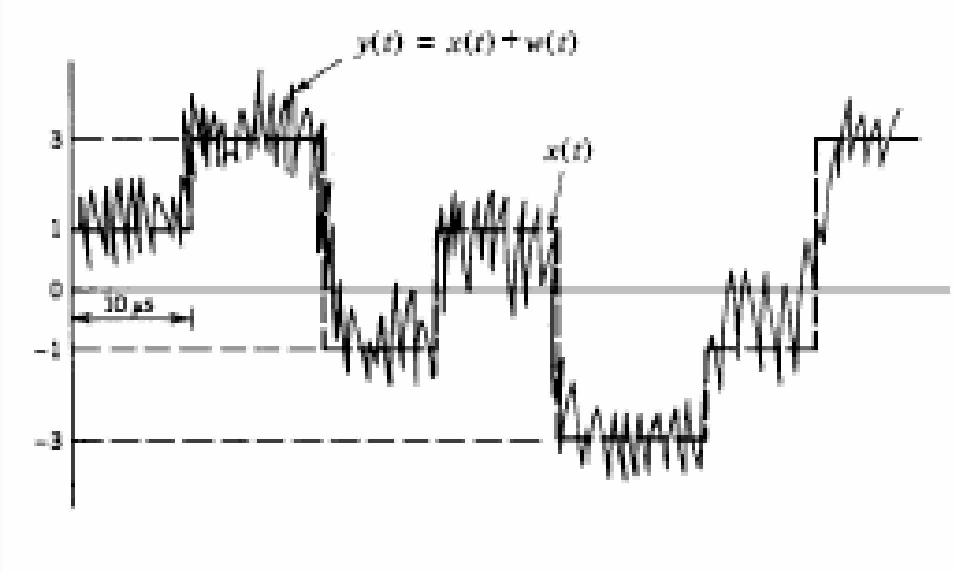
$s_i(t)$

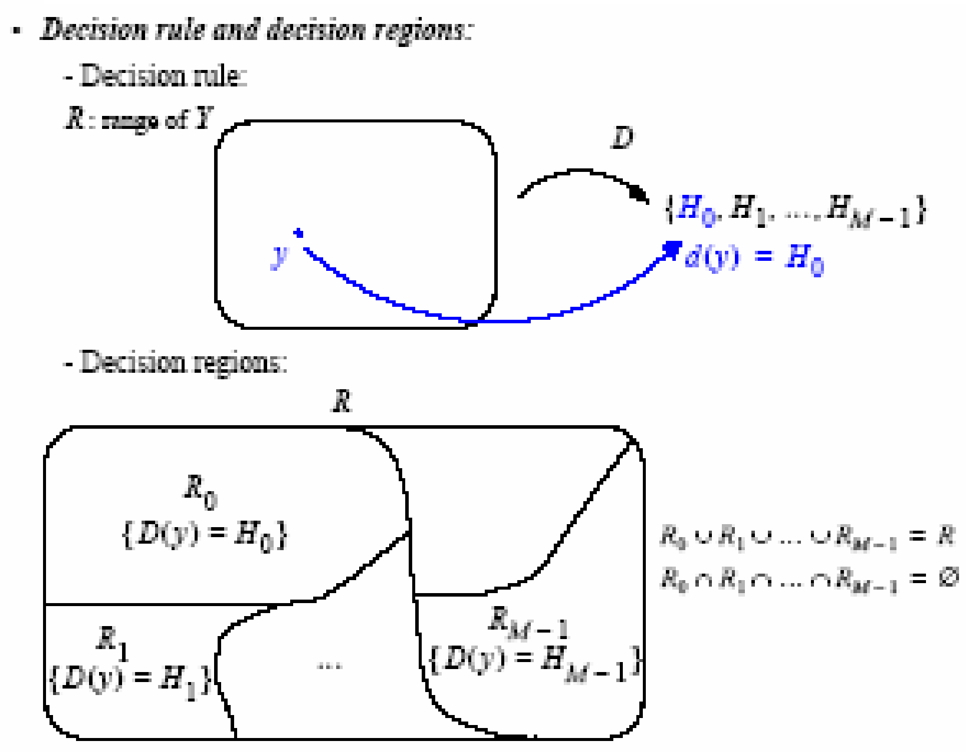
- MAP decision rule:

$$\int_{T_s}^{T_e} y(t)(s_1(t)-s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\left(\frac{P(H_0)}{P(H_1)}\right)+\frac{1}{2}\left(E_{\mathbf{s}_1}-E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \int_{T_s}^{T_e} (s_i(t))^2\,dt$$

4.3 M-ary Detection

- Multiple hypothesis testing: $H_0, H_1, \ldots H_{M-1}$





MAP for M-ary Decision

- MAP decision rule:

$$select \quad H_i \quad if \quad P(H_i \mid y) \geq P(H_j \mid y) \quad for \quad any \quad j = 0,1,\cdots,M-1$$

$$or$$

$$select \quad H_i \quad if \quad \frac{f(y \mid H_i)}{f(y \mid H_j)} \geq \frac{P(H_j)}{P(H_i)} \quad for \quad any \quad j = 0,1,\cdots,M-1$$

MAP for M-ary Decision: Specialities

- MAP decision rule for time-limited discrete-time signals:

$$select \quad H_i \quad if \quad \mathbf{y}^{\mathbf{T}}\mathbf{s_i} + \sigma^2 \ln(P(H_i)) - \frac{1}{2}E_{\mathbf{s_i}} \geq \mathbf{y}^{\mathbf{T}}\mathbf{s}_j + \sigma^2 \ln(P(H_j)) - \frac{1}{2}E_{\mathbf{s}_j} \quad for \quad any \quad j = 0,1,\cdots,M-1$$

- Further with uniform"a priori" pdf, i.e.,

$$P(H_0)=P(H_1)=\ldots=P(H_{M-1})=1/M$$

$$select \quad H_i \quad if \quad \mathbf{y}^{\mathbf{T}}\mathbf{s_i} - \frac{1}{2}E_{\mathbf{s_i}} \geq \mathbf{y}^{\mathbf{T}}\mathbf{s}_j - \frac{1}{2}E_{\mathbf{s}_j} \quad for \quad any \quad j = 0,1,\cdots,M-1$$

MM4 Exercise

Page 374:

- Problem 6.13
- Problem 6.14

Self-Study Presentation I

PLAN

- MM5 Wiener filter I – Least Mea Squared error esitmation: Johnny, Uffe & Joseph
- MM6 Wiener filter II – Wiener filters: Lars, Jacob & Michel
- MM7 Kalman filter I – Ole, Sannina & Morten
- MM8 Kalman filter II – (?)
- MM9 Spectrum esitmation I – Daniel(?)
- MM10 Spectrum estimation II

Self-Study Presentation II

Material

- **Textbook + other materials found by your selves**

Slides

- **Use Ole's slides (AAU) or prepare by yourselves**

Time

- **It's up to the group, no more than 1.5 hours per presentation**

- **Exercises (?)**