# MM4. Signal Detection (Part Two)

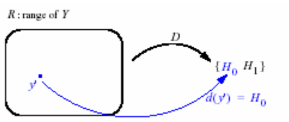
**Reading page: Chapt 6, pp.352-361, 366-370**

- Explain MM3 exercise
- 4.1 Binary detection of discrete-time signals
- 4.2 Binary detection of continuous-time signals
- 4.3 M-ary detection
- Remark of self-study presentation



# What have we talked through MM3?






# 3.1 Hypothesis Testing

In hypothesis testing, a **decision** is made based on the **observation** of a random variable as to which of several **hypotheses** to accept



# 3.2 Decision Rules

- Maximum "a posteriori" decision rule
- Bayes decision rule – Costs of errors
- Minmax rule and Neyman-Pearson rule



# Maximum "a posteriori" (MAP) Rule

- MAP decision rule:

$$f(y\,|\,H_1)P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} f(y\,|\,H_0)P(H_0)$$

- Prior distribution $P(H_i)$, i=1,2
- Likelihood ratio L(y):

$$L(y) = \frac{f(y\,|\,H_1)}{f(y\,|\,H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

$$l(y) = \ln(L(y))$$

$$l(y) = \ln\left(\frac{f(y\,|\,H_1)}{f(y\,|\,H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$



# Bayes' Decision Rule – Cost of Errors

- Motivation: cost of wrong decisions

***Cost function:***
In many engineering branches costs have to be taken into account depending on the decision and the true hypothesis.

| Decision $D$ | True hypothesis $H_0$ | True hypothesis $H_1$ |
|---|---|---|
| $H_0$ | $C_{00}$ | $C_{01}$ |
| $H_1$ | $C_{10}$ | $C_{11}$ |

Usually, the cost of making a wrong decision is higher than that of making a correct decision:

$C_{10} \geq C_{00}$ and $C_{01} \geq C_{11}$.

## Bayes' Decision Rule

- Average cost:

$$\overline{C} = C_{00}P[D=H_0|H_0]P[H_0] + C_{10}P[D=H_1|H_0]P[H_0] + \\ + C_{01}P[D=H_0|H_1]P[H_1] + C_{11}P[D=H_1|H_1]P[H_1]$$

- Bayes' decision rule: minimize the average cost

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)(C_{10}-C_{00})}{P(H_1)(C_{01}-C_{11})}$$



## Explain MM3 Exercise!



## Q-function for Gaussian Distribution

- Gaussian pdf:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X{}^2}} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X{}^2}\right\}$$

- Probability distribution:

$$P(X>a) = \int_a^\infty f_X(x)dx = \int_a^\infty \frac{1}{\sqrt{2\pi\sigma_X{}^2}} \exp\left\{-\frac{(x-\mu_X)^2}{2\sigma_X{}^2}\right\}dx$$

$$P(X>a) = \int_{(a-\mu_X)/\sigma_X}^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\}dx$$

- Q-function:

$$Q(y) = \frac{1}{\sqrt{2\pi}} \int_y^\infty \exp\left\{-\frac{z^2}{2}\right\}dz$$



## 4.1 Binary detection of discrete-time signals

-**Binary detection with multiple observations**

-***Time-limited case***

-*Time-infinite but finite energy case*



## *Time-limited:* Assumptions

- Singal model: Y(n)=x(n)+W(n), n=0,1,...N-1

$$x(n) = \begin{cases} s_0(n) & \text{under hypothesis } H_0 \\ s_1(n) & \text{under hypothesis } H_1 \end{cases}$$

- W(n) is a white Guaussian noise:

$$W(n) \quad \text{is a Gaussian process}$$
$$E\{W(n)\} = 0$$
$$R_{WW}(k) = \sigma^2\delta(k)$$



## *Time-limited:* Notations

- Vector expressions:

$$\mathbf{Y} = [Y_0 \quad Y_1 \quad \cdots \quad Y_{N-1}]^T \quad \mathbf{y} = [y_0 \quad y_1 \quad \cdots \quad y_{N-1}]^T$$
$$\mathbf{S}_0 = [s_{00} \quad s_{01} \quad \cdots \quad s_{0N-1}]^T \quad \mathbf{S}_1 = [s_{10} \quad s_{11} \quad \cdots \quad s_{1N-1}]^T$$
$$\mathbf{W} = [W_0 \quad W_1 \quad \cdots \quad W_{N-1}]^T$$

- Considered signal in vector form:

$$\mathbf{Y} = \mathbf{X} + \mathbf{W}$$

## *Time-limited:* **PDF functions**

- Pdf function of **W:**

$$f(\mathbf{w}) = \prod_{n=0}^{N-1} f(w_n) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{w_n^{\ 2}}{2\sigma^2}\}$$

$$f(\mathbf{w}) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left\{-\frac{\|\mathbf{w}\|^2}{2\sigma^2}\right\} \quad \|\mathbf{w}\|^2 = \sum_{n=0}^{N-1} w_n^{\ 2}$$

- Conditional pdf functions:

$$f(\mathbf{y} \mid H_i) = f(\mathbf{w})|_{\mathbf{w}=\mathbf{y}-\mathbf{s_i}}$$

$$f(\mathbf{y} \mid H_i) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^N \exp\left\{-\frac{\|\mathbf{y}-\mathbf{s_i}\|^2}{2\sigma^2}\right\} \quad \|\mathbf{y}-\mathbf{s_i}\|^2 = \sum_{n=0}^{N-1} (y_n - s_{in})^2 \quad i = 1,2$$



## *Time-limited:* **Likelihood Ratios**

- Likelihood ratio:

$$L(\mathbf{y}) = \frac{f(\mathbf{y} \mid H_1)}{f(\mathbf{y} \mid H_0)} = \frac{\exp\left\{-\frac{\|\mathbf{y}-\mathbf{s_1}\|^2}{2\sigma^2}\right\}}{\exp\left\{-\frac{\|\mathbf{y}-\mathbf{s}_0\|^2}{2\sigma^2}\right\}} = \exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right]\right\}$$

- Loglikelihood ratio:

$$l(\mathbf{y}) = \ln\left(\frac{f(\mathbf{y} \mid H_1)}{f(\mathbf{y} \mid H_0)}\right) = \ln(\exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right]\right\})$$

$$l(\mathbf{y}) = \frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s_1}\|^2\right] = \frac{1}{\sigma^2}\left[\mathbf{y}^{\mathbf{T}}(\mathbf{s_1}-\mathbf{s_0}) + \frac{1}{2}(\|\mathbf{s}_0\|^2 - \|\mathbf{s}_1\|^2)\right]$$



## *Time-limited:* **Decision Rules**

- Decision rule:

$$l(\mathbf{y}) = \frac{1}{\sigma^2}\left[\mathbf{y}^T(\mathbf{s_1}-\mathbf{s_0}) + \frac{1}{2}\left(\|\mathbf{s_0}\|^2 - \|\mathbf{s}_1\|^2\right)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\gamma)$$

$$\mathbf{y}^T(\mathbf{s_1}-\mathbf{s_0}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}^{\ 2}$$

$$\sum_{n=0}^{N-1} y_n(s_{1n} - s_{0n}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right)$$

- MAP decision rule:

$$\mathbf{y}^T(\mathbf{s_1}-\mathbf{s_0}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{P(H_0)}{P(H_1)}\right) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}^{\ 2}$$



# 4.1 Binary detection of discrete-time signals

-**Binary detection with multiple observations**

-***Time-limited case***

-***Time-infinite but finite energy case***



## *Finite-energy:* **Assumptions**

- Singal model: Y(n)=x(n)+W(n), n=...,-2.-1.0,1,...N-1,....

$$x(n) = \begin{cases} s_0(n) & \text{under hypothesis } H_0 \\ s_1(n) & \text{under hypothesis } H_1 \end{cases}$$

$$E_{s_i} = \sum_{n=-\infty}^{\infty} s_{in}^{\ 2} = \|s_{in}\|^2 < \infty$$

- W(n) is a white Guaussian noise:

$$W(n) \quad \text{is a Gaussian process}$$
$$E\{W(n)\} = 0$$
$$R_{WW}(k) = \sigma^2 \delta(k)$$



## *Finite-energy:* **Notations**

- Vector expressions:

$$\mathbf{Y} = [\cdots \; Y_0 \; Y_1 \; \cdots \; Y_{N-1} \; \cdots]^T \quad \mathbf{y} = [\cdots \; y_0 \; y_1 \; \cdots \; y_{N-1} \cdots \;]^T$$
$$\mathbf{S}_0 = [\cdots \; s_{00} \; s_{01} \; \cdots \; s_{0N-1} \cdots \;]^T \quad \mathbf{S}_1 = [\cdots \; s_{10} \; s_{11} \; \cdots \; s_{1N-1} \cdots \;]^T$$
$$\mathbf{W} = [\cdots \; W_0 \; W_1 \; \cdots \; W_{N-1} \cdots \;]^T$$

- Considered signal in vector form:

$$\mathbf{Y} = \mathbf{X} + \mathbf{W}$$

## *Finite-energy:* **PDF functions**

- Pdf function of **W**:

$$f(\mathbf{w}) = \prod_{n=-\infty}^{\infty} f(w_n) = \prod_{n=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{w_n^{\ 2}}{2\sigma^2}\}$$

- Conditional pdf functions:

$$f(\mathbf{y} \mid H_i) = f(\mathbf{w})|_{\mathbf{w}=\mathbf{y}-\mathbf{s}_i}$$

$$f(\mathbf{y} \mid H_i) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{\infty} \exp\left\{-\frac{\|\mathbf{y}-\mathbf{s}_i\|^2}{2\sigma^2}\right\} \quad \|\mathbf{y}-\mathbf{s}_i\|^2 = \sum_{n=-\infty}^{\infty} (y_n - s_{in})^2 \quad i=1,2$$



## *Finite-energy:* **Likelihood Ratios**

- Likelihood ratio:

$$L(\mathbf{y}) = \frac{f(\mathbf{y} \mid H_1)}{f(\mathbf{y} \mid H_0)} = \frac{\exp\left\{-\frac{\|\mathbf{y}-\mathbf{s}_1\|^2}{2\sigma^2}\right\}}{\exp\left\{-\frac{\|\mathbf{y}-\mathbf{s}_0\|^2}{2\sigma^2}\right\}} = \exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s}_1\|^2\right]\right\}$$

- Loglikelihood ratio:

$$l(\mathbf{y}) = \ln\left(\frac{f(\mathbf{y} \mid H_1)}{f(\mathbf{y} \mid H_0)}\right) = \ln(\exp\left\{\frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s}_1\|^2\right]\right\})$$

$$l(\mathbf{y}) = \frac{1}{2\sigma^2}\left[\|\mathbf{y}-\mathbf{s}_0\|^2 - \|\mathbf{y}-\mathbf{s}_1\|^2\right] = \frac{1}{\sigma^2}\left[\mathbf{y}^{\mathbf{T}}(\mathbf{s}_1 - \mathbf{s}_0) + \frac{1}{2}(\|\mathbf{s}_0\|^2 - \|\mathbf{s}_1\|^2)\right]$$



## *Finite-energy:* **Decision Rules**

- Decision rule:

$$l(\mathbf{y}) = \frac{1}{\sigma^2}\left[\mathbf{y}^T(\mathbf{s}_1 - \mathbf{s}_0) + \frac{1}{2}\left(\|\mathbf{s}_0\|^2 - \|\mathbf{s}_1\|^2\right)\right] \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\gamma)$$

$$\mathbf{y}^T(\mathbf{s}_1 - \mathbf{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}^{\ 2}$$

$$\sum_{n=-\infty}^{\infty} y_n(s_{1n} - s_{0n}) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right)$$

- MAP decision rule:

$$\mathbf{y}^T(\mathbf{s}_1 - \mathbf{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \sigma^2 \ln\left(\frac{P(H_0)}{P(H_1)}\right) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \sum_{n=0}^{N-1} s_{in}^{\ 2}$$



# 4.2 Binary detection of continuous-time signals

- **Binary detection with continuous observations**
- ***Bandlimited case***
- ***Time-infinite but finite energy case***



## *Bandlimited:* **Assumptions**

- Singal model: Y(t)=x(t)+W(t),

$$x(t) = \begin{cases} s_0(t) & \text{under hypothesis } H_0 \\ s_1(t) & \text{under hypothesis } H_1 \end{cases}$$

$$E_{s_i} = \int_{-\infty}^{\infty} (s_i(t))^2 dt < \infty \quad \text{and} \quad s_i(t) \text{ is band limited} \quad i=1,2$$

- W(t) is a white Guaussian noise:

$$W(t) \quad \text{is a Gaussian process}$$

$$E\{W(t)\} = 0$$

$$R_{WW}(\tau) = \frac{N_0}{2}\delta(\tau) \quad S_{WW}(f) = \frac{N_0}{2}$$



## *Bandlimited:* **Decision Rules**

- Decision rule:

$$\int_{-\infty}^{\infty} y(t)(s_1(t) - s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln(\gamma) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \int_{-\infty}^{\infty} (s_i(t))^2 dt$$

- MAP decision rule:

$$\int_{-\infty}^{\infty} y(t)(s_1(t) - s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\left(\frac{P(H_0)}{P(H_1)}\right) + \frac{1}{2}\left(E_{\mathbf{s}_1} - E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \int_{-\infty}^{\infty} (s_i(t))^2 dt$$

## *Timelimited:* Decision Rules

- Time-limited but possibly bandwidth unlimited finite-energy signals
- Decision rule:

$$\int_{-T_s}^{T_e} y(t)(s_1(t)-s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln(\gamma)+\frac{1}{2}\left(E_{\mathbf{s}_1}-E_{\mathbf{s}_0}\right) \qquad \overset{s_i(t)}{E_{\mathbf{s}_i}} = \|\mathbf{s}_i\|^2 = \int_{T_s}^{T_e}(s_i(t))^2 dt$$

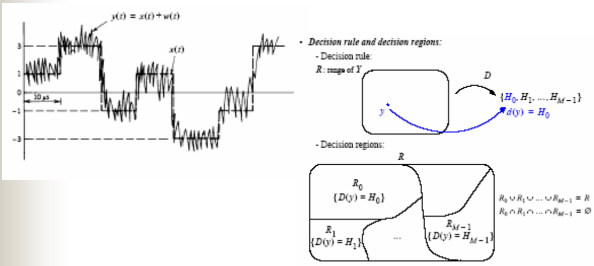
- MAP decision rule:

$$\int_{T_s}^{T_e} y(t)(s_1(t)-s_0(t))dt \underset{H_0}{\overset{H_1}{\gtrless}} \frac{N_0}{2}\ln\left(\frac{P(H_0)}{P(H_1)}\right)+\frac{1}{2}\left(E_{\mathbf{s}_1}-E_{\mathbf{s}_0}\right) \qquad E_{\mathbf{s}_i} = \|\mathbf{s}_i\|^2 = \int_{T_s}^{T_e}(s_i(t))^2 dt$$



## 4.3 M-ary Detection

- Multiple hypothesis testing: $H_0, H_1, \ldots H_{M-1}$






## MAP for M-ary Decision

- MAP decision rule:

$$select \quad H_i \quad if \quad P(H_i \mid y) \geq P(H_j \mid y) \quad for \quad any \quad j = 0,1,\cdots,M-1$$

or

$$select \quad H_i \quad if \quad \frac{f(y \mid H_i)}{f(y \mid H_j)} \geq \frac{P(H_j)}{P(H_i)} \quad for \quad any \quad j = 0,1,\cdots,M-1$$



## MAP for M-ary Decision: Specialities

- MAP decision rule for time-limited discrete-time signals:

$$select \quad H_i \quad if \quad \mathbf{y}^T\mathbf{s}_i + \sigma^2\ln(P(H_i)) - \frac{1}{2}E_{\mathbf{s}_i} \geq \mathbf{y}^T\mathbf{s}_j + \sigma^2\ln(P(H_j)) - \frac{1}{2}E_{\mathbf{s}_j} \quad for \quad any \quad j = 0,1,\cdots,M-1$$

- Further with uniform”a priori” pdf, i.e.,

$P(H_0)=P(H_1)=\ldots=P(H_{M-1})=1/M$

$$select \quad H_i \quad if \quad \mathbf{y}^T\mathbf{s}_i - \frac{1}{2}E_{\mathbf{s}_i} \geq \mathbf{y}^T\mathbf{s}_j - \frac{1}{2}E_{\mathbf{s}_j} \quad for \quad any \quad j = 0,1,\cdots,M-1$$



## MM4 Exercise

Page 374:

- Problem 6.13
- Problem 6.14



## Self-Study Presentation I

**PLAN**

- MM5 Wiener filter I – Least Mea Squared error esitmation: Johnny, Uffe & Joseph
- MM6 Wiener filter II – Wiener filters: Lars, Jacob & Michel
- MM7 Kalman filter I – Ole, Sannina & Morten
- MM8 Kalman filter II – (?)
- MM9 Spectrum esitmation I – Daniel(?)
- MM10 Spectrum estimation II

# Self-Study Presentation II

**Material**

- **Textbook + other materials found by your selves**

**Slides**

- **Use Ole's slides (AAU) or prepare by yourselves**

**Time**

- **It's up to the group, no more than 1.5 hours per presentation**

- **Exercises (?)**