MM2 Robust Analysis and Synthesis

Reference

Some materials are based on the book: Essentials Of Robust Control, Kemin Zhou, John C. Doyle, Published September, 1997 by Prentice Hall

Web: http://www.ee.lsu.edu/kemin/essentials.htm

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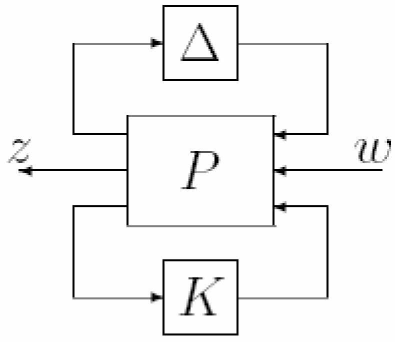
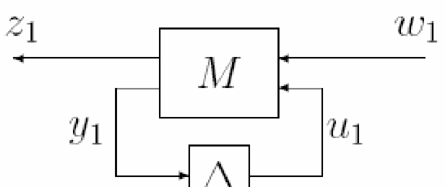
1.1 General framework for robust control analysis and synthesis

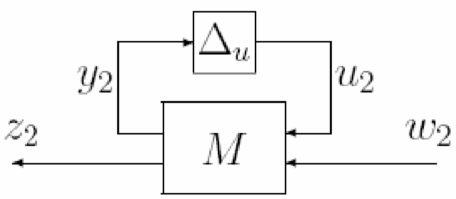


$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) & P_{13}(s) \\ P_{21}(s) & P_{22}(s) & P_{23}(s) \\ P_{31}(s) & P_{32}(s) & P_{33}(s) \end{bmatrix}$$

$$\begin{aligned} z &= \mathcal{F}_u\left(\mathcal{F}_\ell(P, K), \Delta\right) w \\ &= \mathcal{F}_\ell\left(\mathcal{F}_u(P, \Delta), K\right) w. \end{aligned}$$

(Q1: Linear fractional transformation)



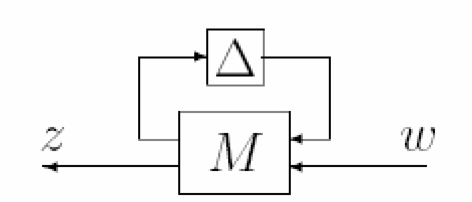


$$\mathcal{F}_\ell(M, \Delta) := M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}$$

$$\mathcal{F}_u(M, \Delta_u) = M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}$$

(Q2: getting the general framework from specific formulation, see black board….)

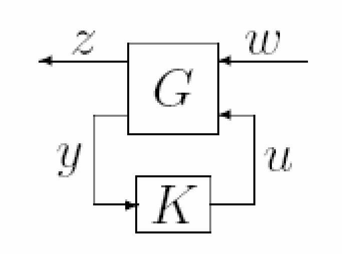
Analysis Framework



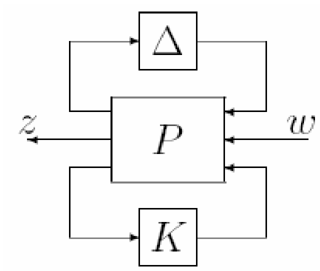
$$M(s) = \mathcal{F}_\ell\left(P(s), K(s)\right) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix},$$

$$z = \mathcal{F}_u(M, \Delta)w = \left[M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}\right] w.$$

Synthesis framework



LQG, H_2, H_infty synthesis



mu synthesis …

1.2 Singular Value Decomposition (SVD)

- What's SVD definition?

Let $A \in \mathbb{F}^{m\times n}$. There exist unitary matrices

$$\begin{aligned} U &= [u_1, u_2, \ldots, u_m] \in \mathbb{F}^{m\times m} \\ V &= [v_1, v_2, \ldots, v_n] \in \mathbb{F}^{n\times n} \end{aligned}$$

such that

$$A = U\Sigma V^*, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}$$

where

$$\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_p \end{bmatrix}$$

and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0, \quad p = \min\{m, n\}.$$

$$\begin{aligned} A^*Av_i &= \sigma_i^2 v_i \\ AA^*u_i &= \sigma_i^2 u_i. \end{aligned}$$

$$\overline{\sigma}(A) = \sigma_{max}(A) = \sigma_1 = \text{the largest singular value of } A;$$

$$\underline{\sigma}(A) = \sigma_{min}(A) = \sigma_p = \text{the smallest singular value of } A$$

- Why talk about SVD?

Singular vectors are good indications of strong/weak input or output directions.

$$\begin{aligned} Av_i &= \sigma_i u_i \\ A^* u_i &= \sigma_i v_i. \end{aligned}$$

Geometrically, the singular values of a matrix A are precisely the lengths of the semi-axes of the hyper-ellipsoid E defined by

$$E = \{y : y = Ax,\ x \in \mathbb{C}^n,\ \|x\| = 1\}.$$

Thus v_1 is the direction in which $\|y\|$ is the largest for all $\|x\| = 1$; while v_n is the direction in which $\|y\|$ is the smallest for all $\|x\| = 1$.

v_1 (v_n) is the *highest (lowest) gain input direction*

u_1 (u_m) is the *highest (lowest) gain observing direction*

$$\overline{\sigma}(A) := \max_{\|x\|=1} \|Ax\|$$

$$\underline{\sigma}(A) := \min_{\|x\|=1} \|Ax\|$$

- Properties of Singular values:

Suppose A and Δ are square matrices. Then

(i) $|\underline{\sigma}(A+\Delta) - \underline{\sigma}(A)| \leq \overline{\sigma}(\Delta)$;

(ii) $\underline{\sigma}(A\Delta) \geq \underline{\sigma}(A)\underline{\sigma}(\Delta)$;

(iii) $\overline{\sigma}(A^{-1}) = \dfrac{1}{\underline{\sigma}(A)}$ if A is invertible.

Let $A \in \mathbb{F}^{m\times n}$ and

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} = \cdots = 0, \quad r \leq \min\{m, n\}.$$

Then

1. $\operatorname{rank}(A) = r$;
2. $\operatorname{Ker}A = \operatorname{span}\{v_{r+1}, \ldots, v_n\}$ and $(\operatorname{Ker}A)^{\perp} = \operatorname{span}\{v_1, \ldots, v_r\}$;
3. $\operatorname{Im}A = \operatorname{span}\{u_1, \ldots, u_r\}$ and $(\operatorname{Im}A)^{\perp} = \operatorname{span}\{u_{r+1}, \ldots, u_m\}$;
4. $A \in \mathbb{F}^{m\times n}$ has a dyadic expansion:

$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^* = U_r \Sigma_r V_r^*$$

where $U_r = [u_1, \ldots, u_r]$, $V_r = [v_1, \ldots, v_r]$, and $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$;

5. $\|A\|_F^2 = \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2$;
6. $\|A\| = \sigma_1$;
7. $\sigma_i(U_0 A V_0) = \sigma_i(A),\ i = 1, \ldots, p$ for any appropriately dimensioned unitary matrices U_0 and V_0;
8. Let $k < r = \operatorname{rank}(A)$ and $A_k := \Sigma_{i=1}^{k} \sigma_i u_i v_i^*$, then

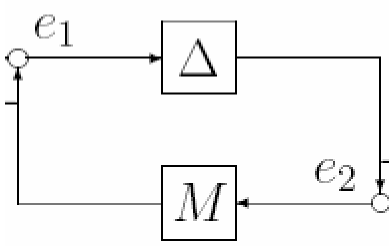
$$\min_{\operatorname{rank}(B)\leq k} \|A - B\| = \|A - A_k\| = \sigma_{k+1}.$$

2. Robust Analysis

2.1 Robust stability

Internal stability (MM1)

Robust stability (Unstructured uncertainties)

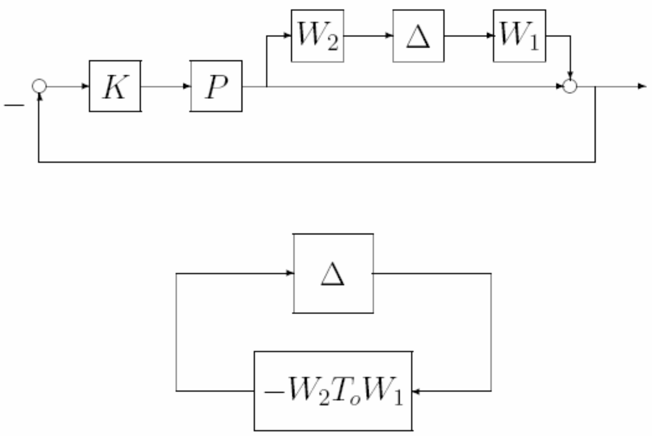


Small Gain Theorem: Suppose $M \in (\mathcal{RH}_\infty)^{p\times q}$. Then the system is well-posed and internally stable for all $\Delta(s) \in \mathcal{RH}_\infty$ with

(a) $\|\Delta\|_\infty \le 1/\gamma$ if and only if $\|M(s)\|_\infty < \gamma$;

(b) $\|\Delta\|_\infty < 1/\gamma$ if and only if $\|M(s)\|_\infty \le \gamma$.

Example:



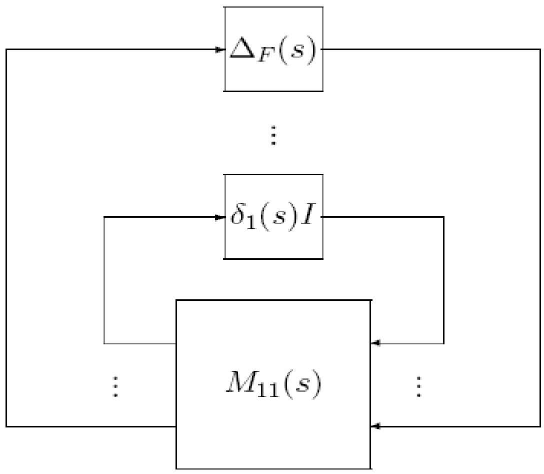
Let $\mathbf{\Pi} = \{(I + W_1\Delta W_2)P : \quad \Delta \in \mathcal{RH}_\infty\}$ and let K stabilize P. Then the closed-loop system is well-posed and internally stable for all $\Delta \in \mathcal{RH}_\infty$ with $\|\Delta\|_\infty < 1$ if and only if $\|W_2T_oW_1\|_\infty \le 1$.

Robust stability (Structured uncertainties)

$$\Delta(s) = \text{diag}\ [\delta_1 I_{r_1}, \ldots, \delta_s I_{r_S}, \Delta_1, \ldots, \Delta_F] \in \mathcal{RH}_\infty$$

$$\|\delta_i\|_\infty < 1 \text{ and } \|\Delta_j\|_\infty < 1.$$

Robust Stability $\Longleftrightarrow$ The following interconnection is stable.



Structured **Singular Value (SSV):**

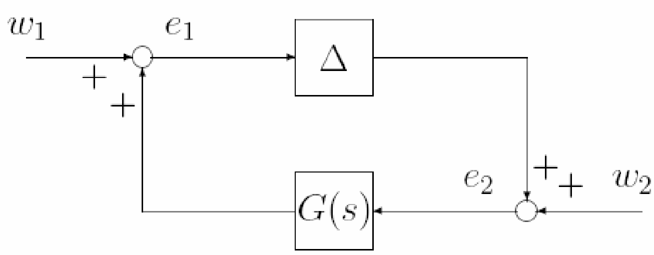
For $M \in \mathbb{C}^{n\times n}$, $\mu_{\mathbf{\Delta}}(M)$ *is defined as*

$$\mu_{\mathbf{\Delta}}(M) := \frac{1}{\min\{\overline{\sigma}(\Delta) : \Delta \in \mathbf{\Delta}, \det(I - M\Delta) = 0\}}$$

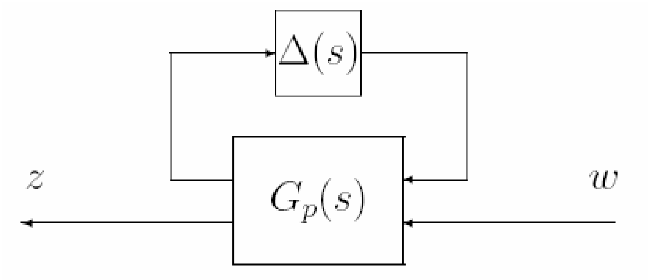
$$\rho(M) \le \mu_{\mathbf{\Delta}}(M) \le \overline{\sigma}(M)$$

Let $\beta > 0$*. The system is well-posed and internally stable for all* $\Delta(\cdot) \in \mathbf{\Delta}$ *with* $\|\Delta\|_\infty < \frac{1}{\beta}$ *if and only if*

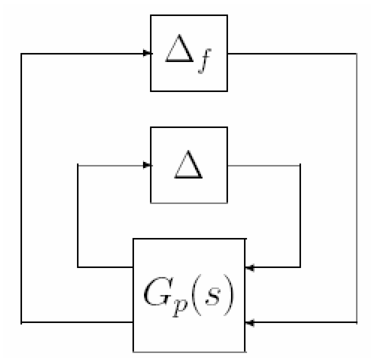
$$\sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta}}(G(j\omega)) \le \beta$$



2.2 Robust Performance (Structured uncertainties)



$$\mathbf{\Delta}_P := \left\{ \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_f \end{bmatrix} : \Delta \in \mathbf{\Delta}, \Delta_f \in \mathbb{C}^{q_2 \times p_2} \right\}$$



Let $\beta > 0$. *For all* $\Delta(s) \in \mathbf{\Delta}$ *with* $\|\Delta\|_\infty < \frac{1}{\beta}$, *the system is well-posed, internally stable, and* $\|F_u(G_p, \Delta)\|_\infty \leq \beta$ *if and only if*

$$\sup_{\omega \in \mathbb{R}} \mu_{\mathbf{\Delta}_P}(G_p(j\omega)) \leq \beta.$$

2.3 Matlab Functions

Matlab Commands (relevant to Robust control toolbox, mu-analysis, LMI toolbox)

$\gg$ **G=pck(A, B, C, D)** % pack the realization in partitioned form

$\gg$ **seesys(G)** % display G in partitioned format

$\gg$ **[A, B, C, D]=unpck(G)** % unpack the system matrix

$\gg$ **G=pck([], [], [], 10)** % create a constant system matrix

$\gg$ **[y, x, t]=step(A, B, C, D, Iu)** % Iu=i (step response of the ith channel)

$\gg$ **[y, x, t]=initial(A, B, C, D, $\mathbf{x_0}$)** % initial response with initial condition x_0

$\gg$ **[y, x, t]=impulse(A, B, C, D, Iu)** % impulse response of the Iuth channel

$\gg$ **[y,x]=lsim(A,B,C,D,U,T)** % U is a length(T) $\times$ column(B) matrix input; T is the sampling points.

$$G_1G_2 \iff \mathbf{mmult}(\mathbf{G_1}, \mathbf{G_2}), \quad \begin{bmatrix} G_1 & G_2 \end{bmatrix} \iff \mathbf{sbs}(\mathbf{G_1}, \mathbf{G_2})$$

$$G_1 + G_2 \iff \mathbf{madd}(\mathbf{G_1}, \mathbf{G_2}), \quad G_1 - G_2 \iff \mathbf{msub}(\mathbf{G_1}, \mathbf{G_2})$$

$$\begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \iff \mathbf{abv}(\mathbf{G_1}, \mathbf{G_2}), \quad \begin{bmatrix} G_1 & \\ & G_2 \end{bmatrix} \iff \mathbf{daug}(\mathbf{G_1}, \mathbf{G_2}),$$

$$G^T(s) \iff \mathbf{transp}(\mathbf{G}), \quad G^{\sim}(s) \iff \mathbf{cjt}(\mathbf{G}), \quad G^{-1}(s) \iff \mathbf{minv}(\mathbf{G})$$

$$\alpha\, G(s) \iff \mathbf{mscl}(\mathbf{G}, \alpha), \quad \alpha \text{ is a scalar.}$$

Consider a mass/spring/damper system as shown in Figure 0.1.

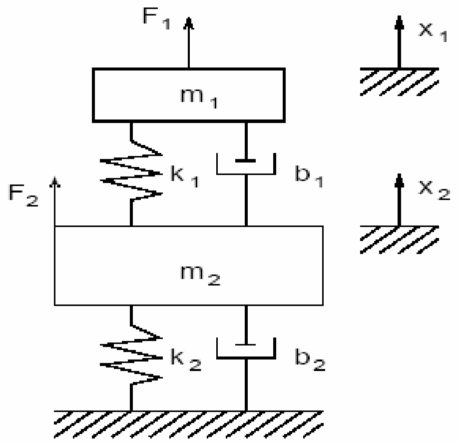


Figure 0.1: A two-mass/spring/damper system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + B \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{m_1} & \frac{k_1}{m_1} & -\frac{b_1}{m_1} & \frac{b_1}{m_1} \\ \frac{k_1}{m_2} & -\frac{k_1+k_2}{m_2} & \frac{b_1}{m_2} & -\frac{b_1+b_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}.$$

Suppose that $G(s)$ is the transfer matrix from (F_1, F_2) to (x_1, x_2); that is,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D = 0,$$

and suppose $k_1 = 1$, $k_2 = 4$, $b_1 = 0.2$, $b_2 = 0.1$, $m_1 = 1$, and $m_2 = 2$ with appropriate units.

≫ **G=pck(A,B,C,D);**

≫ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error ≤ 0.0001

≫ **w=logspace(-1,1,200);** % **200** points between $1 = 10^{-1}$ and $10 = 10^{1}$;

≫ **Gf=frsp(G,w);** % computing frequency response;

≫ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

≫ **vplot('liv, lm', s),** **grid** % plot both singular values and grid.

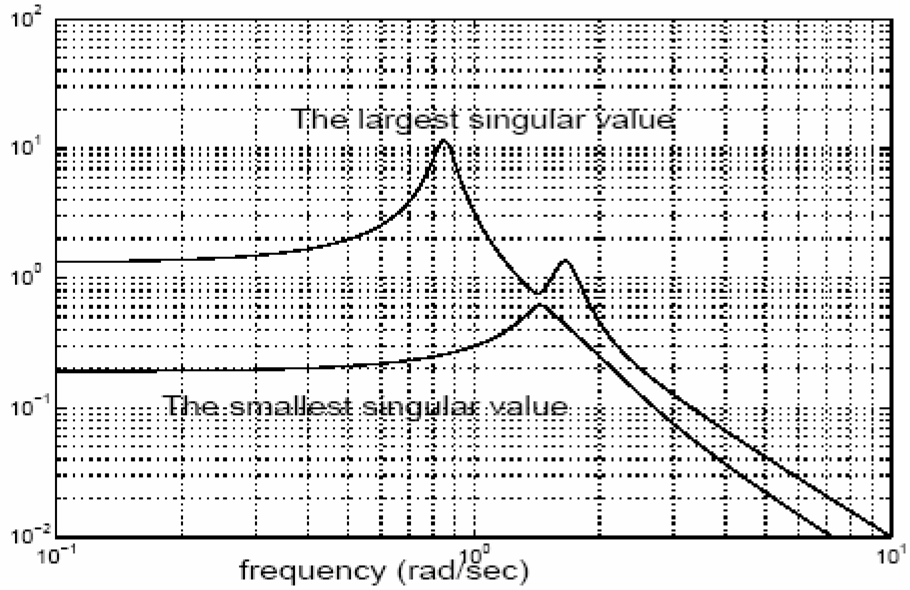


Figure 0.2: $\|G\|_\infty$ is the peak of the largest singular value of $G(j\omega)$

Consider a two-by-two transfer matrix

$$G(s) = \begin{bmatrix} \dfrac{10(s+1)}{s^2+0.2s+100} & \dfrac{1}{s+1} \\ \dfrac{s+2}{s^2+0.1s+10} & \dfrac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}.$$

A state-space realization of G can be obtained using the following MATLAB commands:

≫ **G11=nd2sys([10,10],[1,0.2,100]);**

≫ **G12=nd2sys(1,[1,1]);**

≫ **G21=nd2sys([1,2],[1,0.1,10]);**

≫ **G22=nd2sys([5,5],[1,5,6]);**

≫ **G=sbs(abv(G11,G21),abv(G12,G22));**

Next, we set up a frequency grid to compute the frequency response of G and the singular values of $G(j\omega)$ over a suitable range of frequency.

≫ **w=logspace(0,2,200);** % **200** points between $1 = 10^0$ and $100 = 10^2$;

≫ **Gf=frsp(G,w);** % computing frequency response;

≫ **[u,s,v]=vsvd(Gf);** % SVD at each frequency;

≫ **vplot('liv, lm', s), grid** % plot both singular values and grid;

≫ **pkvnorm(s)** % find the norm from the frequency response of the singular values.

The singular values of $G(j\omega)$ are plotted in Figure 0.3, which gives an estimate of $\|G\|_\infty \approx 32.861$. The state-space bisection algorithm described previously leads to $\|G\|_\infty = 50.25 \pm 0.01$ and the corresponding MATLAB command is

$\gg$ **hinfnorm(G,0.0001)** or **linfnorm(G,0.0001)** % relative error < 0.0001.

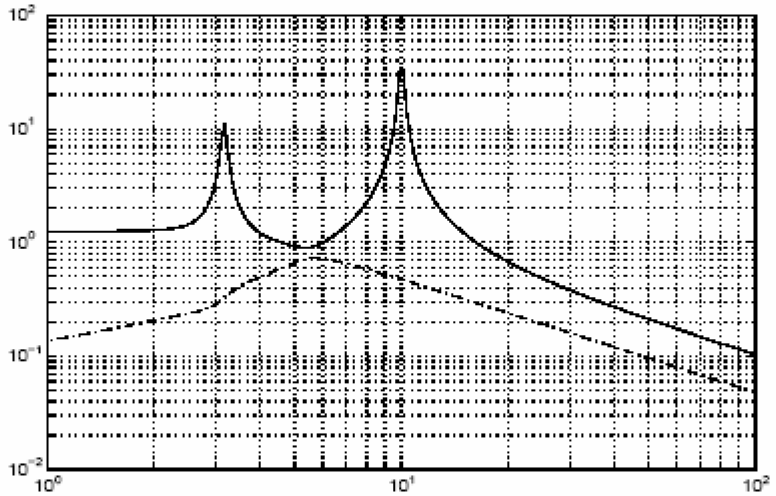


Figure 0.3: The largest and the smallest singular values of $G(j\omega)$

The preceding computational results show clearly that the graphical method can lead to a wrong answer for a lightly damped system if the frequency grid is not sufficiently dense. Indeed, we would get $\|G\|_\infty \approx 43.525, 48.286$ and 49.737 from the graphical method if $400, 800$, and 1600 frequency points are used, respectively.

H_inf Control of an Aircraft (in K. Zhou's Book)

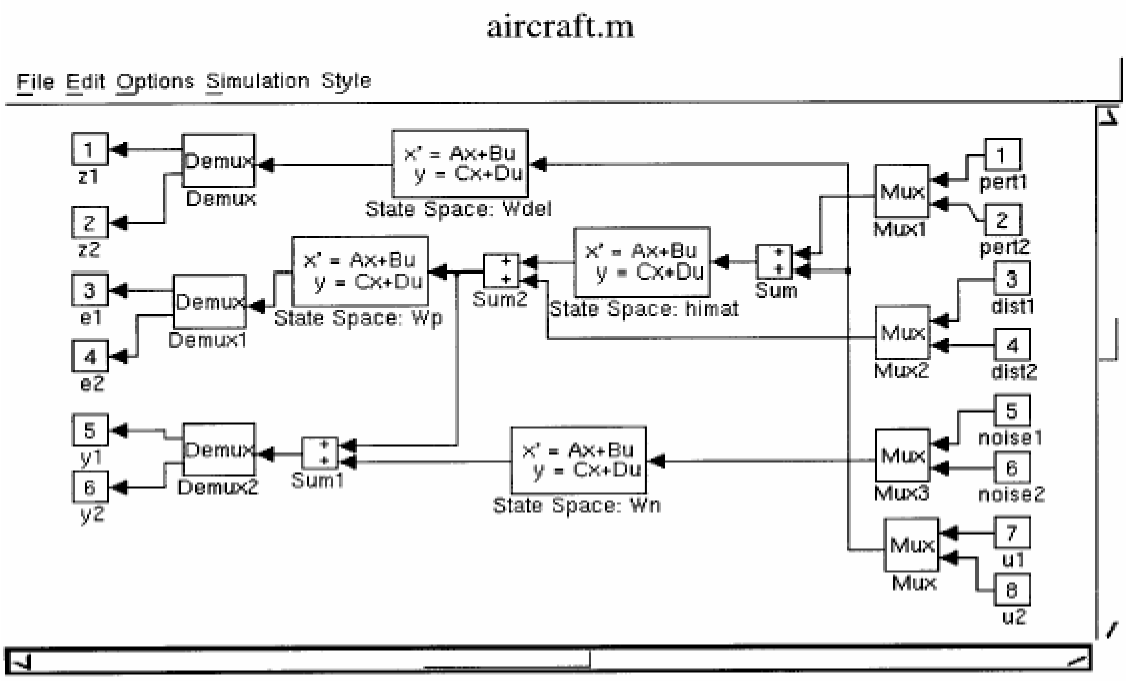
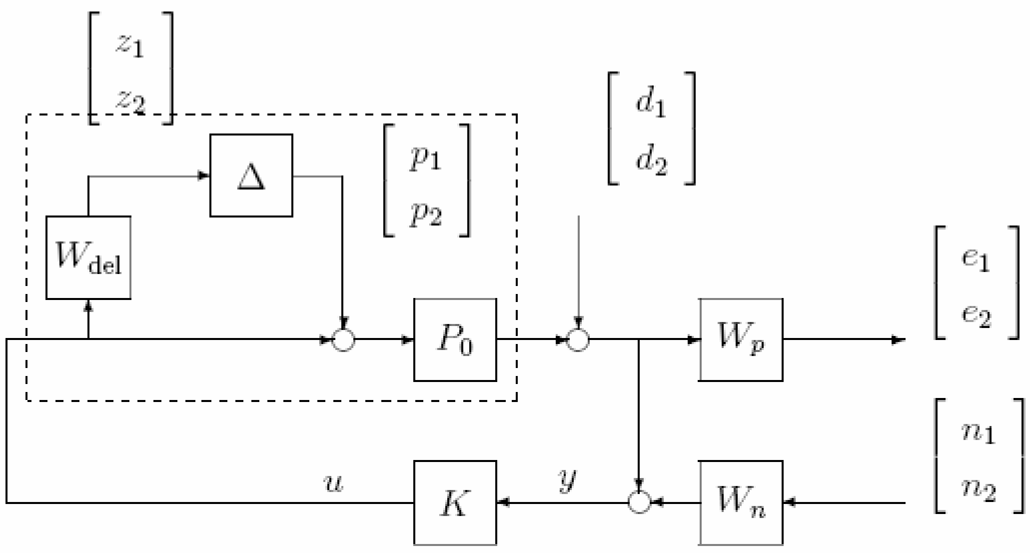


Figure 0.20: SIMULINK block diagram for HIMAT (aircraft.m)



$$W_{\text{del}} = \begin{bmatrix} \frac{50(s+100)}{s+10000} & 0 \\ 0 & \frac{50(s+100)}{s+10000} \end{bmatrix}, \quad W_p = \begin{bmatrix} \frac{0.5(s+3)}{s+0.03} & 0 \\ 0 & \frac{0.5(s+3)}{s+0.03} \end{bmatrix},$$

$$W_n = \begin{bmatrix} \frac{2(s+1.28)}{s+320} & 0 \\ 0 & \frac{2(s+1.28)}{s+320} \end{bmatrix},$$

$$P_0 = \left[\begin{array}{cccc|cc} -0.0226 & -36.6 & -18.9 & -32.1 & 0 & 0 \\ 0 & -1.9 & 0.983 & 0 & -0.414 & 0 \\ 0.0123 & -11.7 & -2.63 & 0 & -77.8 & 22.4 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 57.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 57.3 & 0 & 0 \end{array}\right]$$

$\gg$ $[\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}] = \mathbf{linmod}('\mathbf{aircraft}')$

$\gg$ $\hat{\mathbf{G}} = \mathbf{pck}(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D});$

$\gg$ $[\mathbf{K}, \mathbf{G_p}, \gamma] = \mathbf{hinfsyn}(\hat{\mathbf{G}}, \mathbf{2}, \mathbf{2}, \mathbf{0}, \mathbf{10}, \mathbf{0.001}, \mathbf{2});$

which gives $\gamma = 1.8612 = \|G_p\|_\infty$, a stabilizing controller K, and a closed loop transfer matrix G_p:

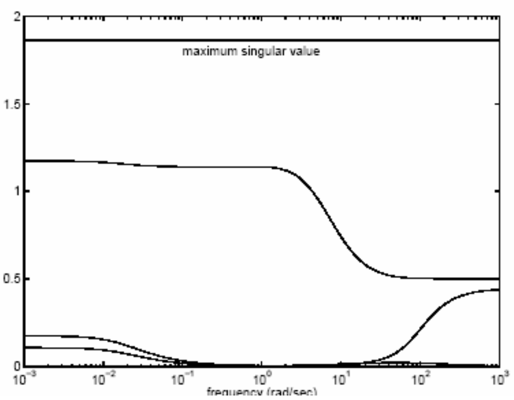
$$\left[\begin{array}{c} z_1 \\ z_2 \\ \hdashline e_1 \\ e_2 \end{array}\right] = G_p(s) \left[\begin{array}{c} p_1 \\ p_2 \\ \hdashline d_1 \\ d_2 \\ n_1 \\ n_2 \end{array}\right], \quad G_p(s) = \begin{bmatrix} G_{p11} & G_{p12} \\ G_{p21} & G_{p22} \end{bmatrix}.$$

$\gg$ $\mathbf{w{=}logspace(\text{-}3,3,300);}$

$\gg$ $\mathbf{Gpf} = \mathbf{frsp}(\mathbf{G_p}, \mathbf{w});$ %

$\gg$ $[\mathbf{u}, \mathbf{s}, \mathbf{v}] = \mathbf{vsvd}(\mathbf{Gpf});$

$\gg$ $\mathbf{vplot}('\mathbf{liv}, \mathbf{m}', \mathbf{s})$



To test the robust stability, we need to compute $\|G_{p11}\|_\infty$*:*

$\gg \mathbf{G_{p11} = sel(G_p, 1:2, 1:2);}$

$\gg \mathbf{norm\_of\_G_{p11} = hinfnorm(G_{p11}, 0.001);}$

$\|G_{P11}\|_\infty = 0.933 < 1$. *So the system is robustly stable.*

To check the robust performance, we shall compute the $\mu_{\Delta_P}(G_p(j\omega))$ *for each frequency with*

$$\Delta_P = \begin{bmatrix} \Delta & \\ & \Delta_f \end{bmatrix}, \ \Delta \in \mathbb{C}^{2\times 2}, \quad \Delta_f \in \mathbb{C}^{4\times 2}.$$

```
>> blk=[2,2;4,2];
>> [bnds,dvec,sens,pvec]=mu(Gpf,blk);
>> vplot('liv, m', vnorm(Gpf), bnds)
>> title('Maximum Singular Value and mu')
>> xlabel('frequency(rad/sec)')
>> text(0.01, 1.7,'maximum singular value')
>> text(0.5, 0.8,'mu bounds')
```