MM2 Robust Analysis and Synthesis

Reference

Some materials are based on the book: Essentials Of Robust Control, Kemin Zhou, John C. Doyle, Published September, 1997 by Prentice Hall

Web: http://www.ee.lsu.edu/kemin/essentials.htm

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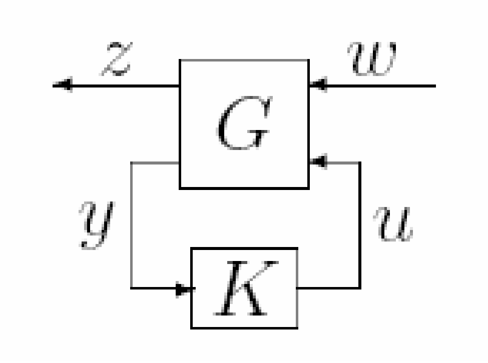
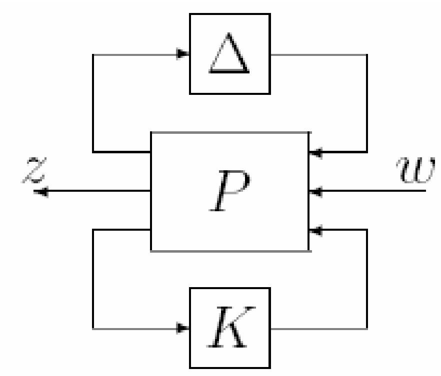
4. Robust control toolbox & mu-analysis toolbox

Synthesis framework

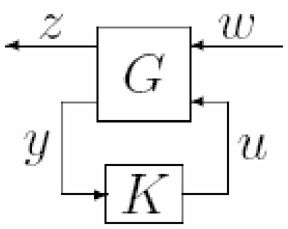


LQG, H_2, H_infty synthesis



mu synthesis …

3.1 H_infty Control Synthesis



$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

(A1) (A, B_2) is stabilizable and (C_2, A) is detectable;

(A2) $D_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$ and $D_{21} = \begin{bmatrix} 0 & I \end{bmatrix}$;

(A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(A4) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

$$R := D_{1\bullet}^* D_{1\bullet} - \begin{bmatrix} \gamma^2 I_{m_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{1\bullet} := [D_{11} \;\; D_{12}]$$

$$\tilde{R} := D_{\bullet 1} D_{\bullet 1}^* - \begin{bmatrix} \gamma^2 I_{p_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{where} \quad D_{\bullet 1} := \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

$$H_\infty := \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_{1\bullet} \end{bmatrix} R^{-1} \begin{bmatrix} D_{1\bullet}^* C_1 & B^* \end{bmatrix}$$

$$J_\infty := \begin{bmatrix} A^* & 0 \\ -B_1 B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1 D_{\bullet 1}^* \end{bmatrix} \tilde{R}^{-1} \begin{bmatrix} D_{\bullet 1} B_1^* & C \end{bmatrix}$$

$$X_\infty := Ric(H_\infty) \qquad Y_\infty := Ric(J_\infty)$$

$$F := \begin{bmatrix} F_{1\infty} \\ F_{2\infty} \end{bmatrix} := -R^{-1}\left[D_{1\bullet}^* C_1 + B^* X_\infty\right]$$

$$L := \begin{bmatrix} L_{1\infty} & L_{2\infty} \end{bmatrix} := -[B_1 D_{\bullet 1}^* + Y_\infty C^*]\tilde{R}^{-1}$$

D, $F_{1\infty}$, and $L_{1\infty}$ are Partitioned as follows:

$$\left[\begin{array}{c|c} & F' \\ \hline L' & D \end{array}\right] = \left[\begin{array}{c|ccc} & F_{11\infty}^* & F_{12\infty}^* & F_{2\infty}^* \\ \hline L_{11\infty}^* & D_{1111} & D_{1112} & 0 \\ L_{12\infty}^* & D_{1121} & D_{1122} & I \\ L_{2\infty}^* & 0 & I & 0 \end{array}\right].$$

There exists a stabilizing controller $K(s)$ such that

$$\|\mathcal{F}_\ell(G,K)\|_\infty < \gamma$$

if and only if

(i) $\gamma > max(\overline{\sigma}[D_{1111}, D_{1112},], \overline{\sigma}[D_{1111}^*, D_{1121}^*])$;

(ii) $H_\infty \in dom(Ric)$ with $X_\infty = Ric(H_\infty) \geq 0$;

(iii) $J_\infty \in dom(Ric)$ with $Y_\infty = Ric(J_\infty) \geq 0$;

(iv) $\rho(X_\infty Y_\infty) < \gamma^2$.

$$K = \mathcal{F}_\ell(M_\infty, Q), \quad Q \in \mathcal{RH}_\infty, \quad \|Q\|_\infty < \gamma$$

where

$$M_\infty = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array}\right]$$

$$\hat{D}_{11} = -D_{1121}D_{1111}^*(\gamma^2 I - D_{1111}D_{1111}^*)^{-1}D_{1112} - D_{1122},$$

$\hat{D}_{12} \in \mathbb{R}^{m_2 \times m_2}$ and $\hat{D}_{21} \in \mathbb{R}^{p_2 \times p_2}$ are any matrices satisfying

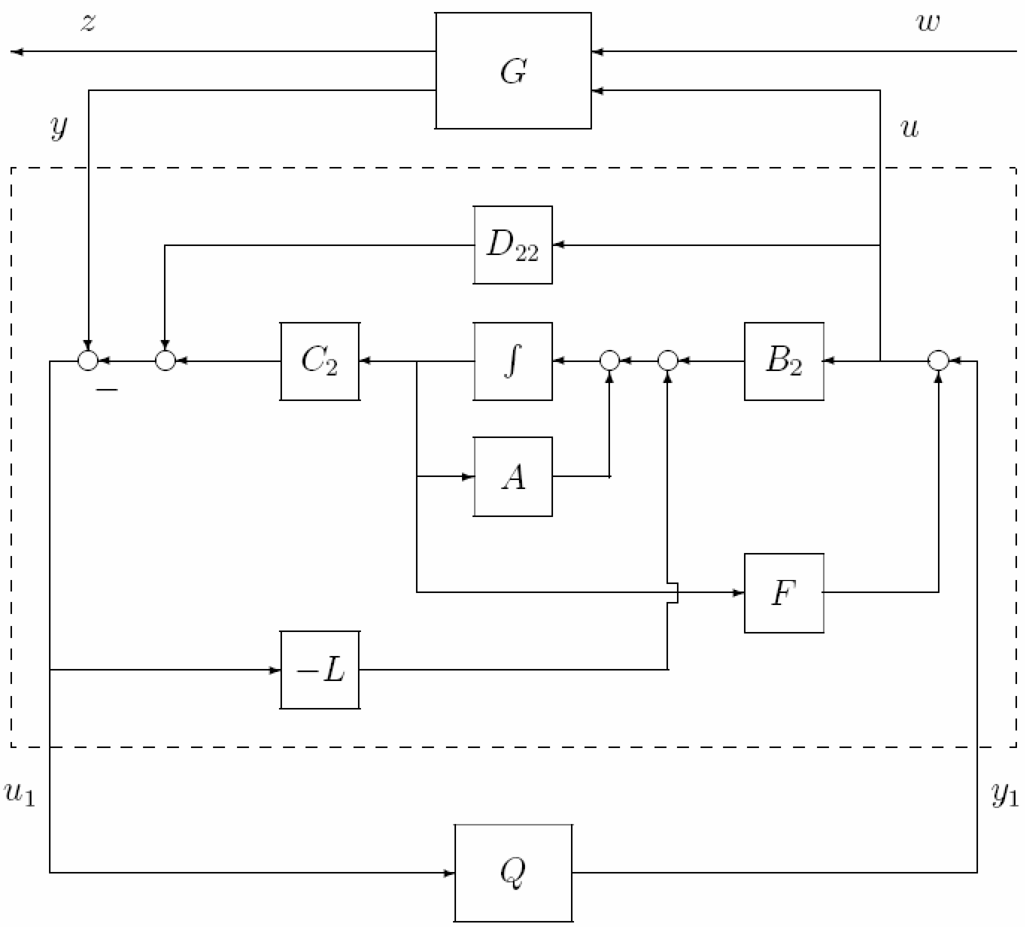
$$\begin{aligned}
\hat{D}_{12}\hat{D}_{12}^* &= I - D_{1121}(\gamma^2 I - D_{1111}^* D_{1111})^{-1} D_{1121}^*, \\
\hat{D}_{21}^*\hat{D}_{21} &= I - D_{1112}^*(\gamma^2 I - D_{1111} D_{1111}^*)^{-1} D_{1112},
\end{aligned}$$

and

$$\begin{aligned}
\hat{B}_2 &= Z_\infty (B_2 + L_{12\infty})\hat{D}_{12}, \\
\hat{C}_2 &= -\hat{D}_{21}(C_2 + F_{12\infty}), \\
\hat{B}_1 &= -Z_\infty L_{2\infty} + \hat{B}_2 \hat{D}_{12}^{-1} \hat{D}_{11}, \\
\hat{C}_1 &= F_{2\infty} + \hat{D}_{11}\hat{D}_{21}^{-1}\hat{C}_2, \\
\hat{A} &= A + BF + \hat{B}_1 \hat{D}_{21}^{-1} \hat{C}_2
\end{aligned}$$

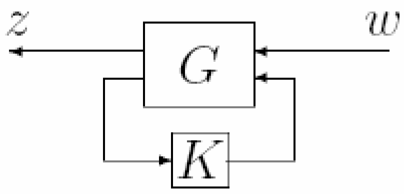
where

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$



Youla Parameterization

3.2 Mu Synthesis



$$\mathcal{F}_\ell(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}.$$

$$\min_K \|\mathcal{F}_\ell(G, K)\|_\mu$$

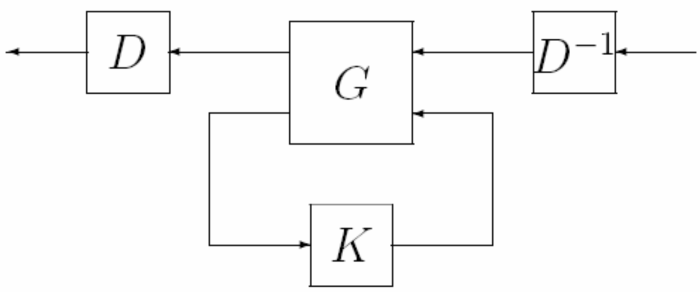
The μ-synthesis is not yet fully solved. But a reasonable approach is to "solve"

$$\min_K \inf_{D, D^{-1} \in \mathcal{H}_\infty} \left\| D\mathcal{F}_\ell(G, K)D^{-1} \right\|_\infty$$

by iteratively solving for K and D, i.e., first minimizing over K with D fixed, then minimizing pointwise over D with K fixed, then again over K, and again over D, etc. This is the so-called *D-K Iteration*.

- Fix D
$$\min_K \left\| D\mathcal{F}_\ell(G, K)D^{-1} \right\|_\infty$$
is a standard $\mathcal{H}_\infty$ optimization problem.

- Fix K
$$\inf_{D, D^{-1} \in \mathcal{H}_\infty} \left\| D\mathcal{F}_\ell(G, K)D^{-1} \right\|_\infty$$
is a standard convex optimization problem and it can be solved pointwise in the frequency domain:
$$\sup_\omega \inf_{D_\omega \in \mathcal{D}} \overline{\sigma} \left[D_\omega \mathcal{F}_\ell(G, K)(j\omega) D_\omega^{-1} \right].$$
Note that when $S = 0$, (no scalar blocks)
$$D_\omega = \text{diag}(d_1^\omega I, \ldots, d_{F-1}^\omega I, I) \in \mathcal{D},$$



D-K Iterations:

(i) Fix an initial estimate of the scaling matrix $D_\omega \in \mathcal{D}$ pointwise across frequency.

(ii) Find scalar transfer functions $d_i(s), d_i^{-1}(s) \in \mathcal{RH}_\infty$ for $i = 1, \ldots, (F-1)$ such that $|d_i(j\omega)| \approx d_i^\omega$.

(iii) Let

$$D(s) = \text{diag}(d_1(s)I, \ldots, d_{F-1}(s)I, I).$$

Construct a state space model for system

$$\hat{G}(s) = \begin{bmatrix} D(s) & \\ & I \end{bmatrix} G(s) \begin{bmatrix} D^{-1}(s) & \\ & I \end{bmatrix}.$$

(iv) Solve an $\mathcal{H}_\infty$-optimization problem to minimize

$$\left\| \mathcal{F}_\ell(\hat{G}, K) \right\|_\infty$$

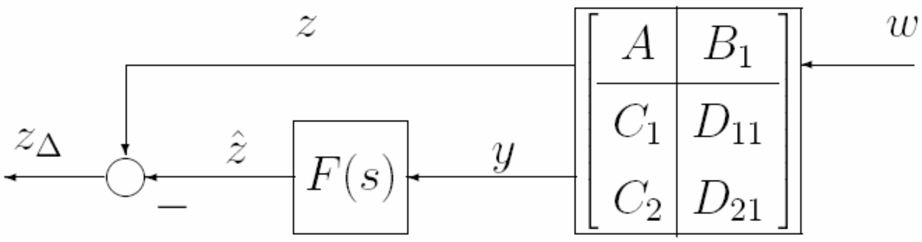
over all stabilizing K's. Denote the minimizing controller by $\hat{K}$.

(v) Minimize $\overline{\sigma}[D_\omega \mathcal{F}_\ell(G, \hat{K}) D_\omega^{-1}]$ over D_ω, pointwise across frequency. The minimization itself produces a new scaling function.

(vi) Compare $\hat{D}_\omega$ with the previous estimate D_ω. Stop if they are close, otherwise, replace D_ω with $\hat{D}_\omega$ and return to step (ii).

The joint optimization of D and K is not convex and the global convergence is not guaranteed, many designs have shown that this approach works very well.

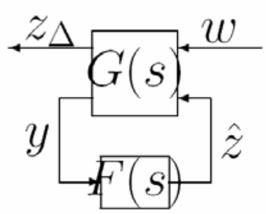
3.3 H_infty Filtering



$\mathcal{H}_\infty$ **Filtering**: Given a $\gamma > 0$, find a causal filter $F(s) \in \mathcal{RH}_\infty$ if it exists such that

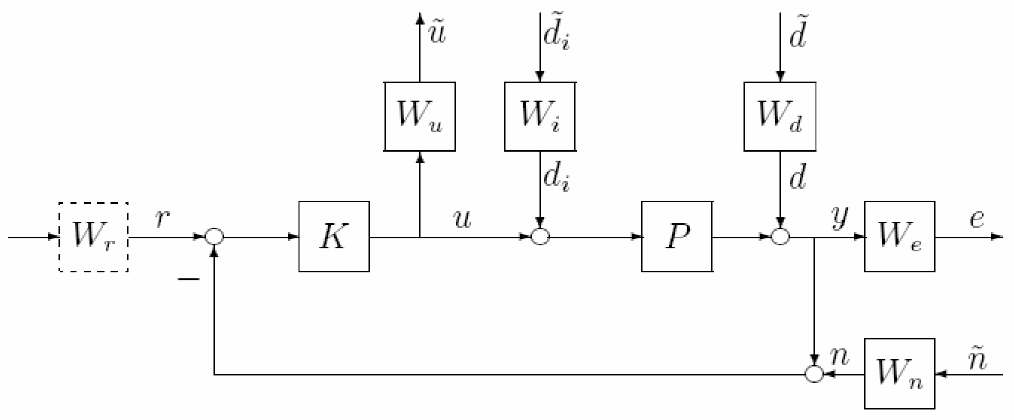
$$J := \sup_{w \in \mathcal{L}_2[0,\infty)} \frac{\|z - \hat{z}\|_2^2}{\|w\|_2^2} < \gamma^2$$

with $\hat{z} = F(s)y$.



$$G(s) = \left[\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & D_{11} & -I \\ C_2 & D_{21} & 0 \end{array}\right]$$

3.4 Weighted H_2 and H_infty Performance



$\mathcal{H}_2$ Performance: Assume $\tilde{d}(t) = \eta\delta(t)$ and $E(\eta\eta^*) = I$

Minimize the expected energy of the error e:

$$E\left\{\|e\|_2^2\right\} = E\left\{\int_0^\infty \|e\|^2\, dt\right\} = \|W_e S_o W_d\|_2^2$$

Include the control signal u in the cost function:

$$E\left\{\|e\|_2^2 + \rho^2 \|\tilde{u}\|_2^2\right\} = \left\| \begin{bmatrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{bmatrix} \right\|_2^2$$

$\mathcal{H}_\infty$ Performance: under worst possible case

$$\sup_{\|\tilde{d}\|_2 \le 1} \|e\|_2 = \|W_e S_o W_d\|_\infty$$

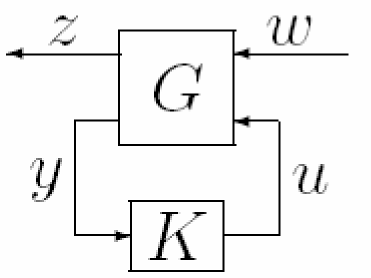
restrictions on the control energy or control bandwidth:

$$\sup_{\|\tilde{d}\|_2 \le 1} \|\tilde{u}\|_2 = \|W_u K S_o W_d\|_\infty$$

Combined cost:

$$\sup_{\|\tilde{d}\|_2 \le 1} \left\{\|e\|_2^2 + \rho^2 \|\tilde{u}\|_2^2\right\} = \left\| \begin{bmatrix} W_e S_o W_d \\ \rho W_u K S_o W_d \end{bmatrix} \right\|_\infty^2$$

3.5 H_2 Optimal Control



$$G(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right]$$

Assumptions:

(i) (A, B_2) is stabilizable and (C_2, A) is detectable;

(ii) D_{12} has full column rank with $\begin{bmatrix} D_{12} & D_\perp \end{bmatrix}$ unitary, and D_{21} has full row rank with $\begin{bmatrix} D_{21} \\ \tilde{D}_\perp \end{bmatrix}$ unitary;

(iii) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all ω;

(iv) $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all ω.

$\mathcal{H}_2$ Problem: *find a stabilizing controller K that minimizes* $\|T_{zw}\|_2$.

There exists a unique optimal controller

$$K_{opt}(s) := \left[\begin{array}{c|c} A + B_2F_2 + L_2C_2 & -L_2 \\ \hline F_2 & 0 \end{array}\right].$$

Moreover, $\min \|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 = \|G_cL_2\|_2^2 + \|C_1G_f\|_2^2$.

$$F_2 := -(B_2^*X_2 + D_{12}^*C_1), \quad L_2 := -(Y_2C_2^* + B_1D_{21}^*)$$

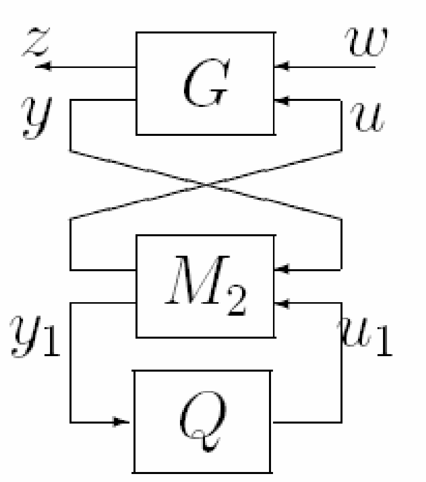
$$X_2(A - B_2D_{12}^*C_1) + (A - B_2D_{12}^*C_1)^*X_2 - X_2B_2B_2^*X_2 + C_1^*D_\perp D_\perp^* C_1 = 0$$

$$Y_2(A - B_1D_{21}^*C_2)^* + (A - B_1D_{21}^*C_2)Y_2 - Y_2C_2^*C_2Y_2 + B_1\tilde{D}_\perp^*\tilde{D}_\perp B_1^* = 0$$

$$G_c(s) := \left[\begin{array}{c|c} A + B_2F_2 & I \\ \hline C_1 + D_{12}F_2 & 0 \end{array}\right], \qquad G_f(s) := \left[\begin{array}{c|c} A + L_2C_2 & B_1 + L_2D_{21} \\ \hline I & 0 \end{array}\right]$$

all stabilizing controllers $K(s) = \mathcal{F}_\ell(M_2, Q), \quad Q \in \mathcal{RH}_\infty$ with

$$M_2(s) = \left[\begin{array}{c|cc} A + B_2F_2 + L_2C_2 & -L_2 & B_2 \\ \hline F_2 & 0 & I \\ -C_2 & I & 0 \end{array}\right].$$



$$\|T_{zw}\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f - QV\|_2^2 = \|G_cB_1\|_2^2 + \|F_2G_f\|_2^2 + \|Q\|_2^2$$

and $Q = 0$ gives the unique optimal control: $K = \mathcal{F}_\ell(M_2, 0)$.

- LQR margin: $\geq 60^o$ phase margin and $\geq 6dB$ gain margin
- LQG or $\mathcal{H}_2$ Controller: No guaranteed margin

4. Matlab Functions for Robust Synthesis

The term control system design refers to the process of synthesizing a feedback control law that meets design specifications in a closed-loop control system. The design methods are iterative, combining parameter selection with analysis, simulation, and insight into the dynamics of the plant. The Robust Control Toolbox provides a set of commands that you can use for a broad range of multivariable control applications, including

- H_2 control design
- H_∞ standard and loop-shaping control design
- H_∞ normalized coprime factor control design
- Mixed H_2/H_∞ control design
- µ-synthesis via D–K iteration
- Sampled-data H_∞ control design

These functions cover both continuous and discrete-time problems. The following table summarizes the H_2 and H_∞ control design commands.

Function	**Description**
`augw`	Augments plant weights for mixed-sensitivity control design
`h2hinfsyn`	Mixed H_2/H_∞ controller synthesis
`h2syn`	H_2 controller synthesis
`hinfsyn`	H_∞ controller synthesis
`loopsyn`	H_∞ loop-shaping controller synthesis
`ltrsyn`	Loop-transfer recovery controller synthesis
`mixsyn`	H_∞ mixed-sensitivity controller synthesis
`ncfsyn`	H_∞ normalized coprime factor controller synthesis
`sdhinfsyn`	Sample-data H_∞ controller synthesis

The following table summarizes μ-synthesis via *D–K* iteration control design commands.

Function	Description
`dksyn`	Synthesis of a robust controller via μ-synthesis
`dkitopt`	Create a `dksyn` options object
`drawmag`	Interactive mouse-based sketching and fitting tool
`fitfrd`	Fit scaling frequency response data with LTI model
`fitmagfrd`	Fit scaling magnitude data with stable, minimum-phase model