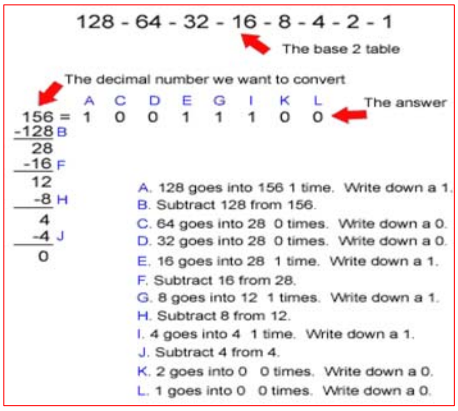
# MM3 Iterative Solution of Equations

- kl.8.15-9.00 review of MM2 and some examples
- kl.9.00 – 10.30 exercise (see notes)
- kl.10.40-11.30 MM3 lecture (I)



## Conversion Methods from Decimal to Binary (MM2)

- Method-1: Comparison with descending powers of two and subtraction
- Division by two with remainder




| | |
|---|---|
| 2)156 | 0 |
| 2)78 | 0 |
| 2)39 | 1 |
| 2)19 | 1 |
| 2)9 | 1 |
| 2)4 | 0 |
| 2)2 | 0 |
| 2)1 | 1 |
| 0 | |

# Conversion Methods from Binary to Decimal (MM2)

- Positional notation method

```
128  64  32  16   8   4   2   1
 1    0   0   1   1   0   1   1
128 + 0 + 0 + 16 + 8 + 0 + 2 + 1 = 155
```

- Doubling method
  1. **1**011001 → *0**2+**1** = 1
  2. **10**11001 → 1*2+**0** = 2
  3. **101**1001 → 2*2+**1** = 5
  4. **1011**001 → 5*2+**1** = 11
  5. **10110**01 → 11*2+**0** = 22
  6. **101100**1 → 22*2+**0** = 44
  7. **1011001** → 44*2+**1** = $89_{10}$



# Conversion methods from Decimal Fractions to Binary (MM2)

- Step 1: Begin with the decimal fraction and multiply by 2. The whole number part of the result is the first binary digit to the right of the point.
- Step 2: Next we disregard the whole number part of the previous result and multiply by 2 once again. The whole number part of this new result is the *second* binary digit to the right of the point. We will continue this process until we get a zero as our decimal part or until we recognize an infinite repeating pattern.

- Infinite Binary Fractions

Example: $0.625_{10}$ = 0.101 (base 2)
$1/10_{10}$ = ??? (base 2)

# Series Expansions (MM2)

*geometric series* :

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots, \qquad |x| < 1$$

*exponential series* :

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad all\ x$$

*trigonometric functions* :

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad all\ x$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad all\ x$$



# Taylor's Theorem (MM2)

**Theorem 1.1 (Taylor's Theorem).** If $f(x)$ has derivatives of order $0, 1, 2, \ldots, n+1$ on the closed interval $[a, b]$, then for any $x$ and $c$ in this interval

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)\,(x-c)^k}{k!} + \frac{f^{(n+1)}(\xi)\,(x-c)^{n+1}}{(n+1)!},$$

where $\xi$ is some number between $x$ and $c$, and $f^k(x)$ is the $k^{\text{th}}$ derivative of $f$ at $x$.

We will use this theorem again and again in this class. The main usage is to approximate a function by the first few terms of its Taylor's series expansion; the theorem then tells us the approximation is "as good" as the final term, also known as the *error term*. That is, we can make the following manipulation:

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(c)\,(x-c)^k}{k!} \right| = \frac{\left|f^{(n+1)}(\xi)\right| |x-c|^{n+1}}{(n+1)!} \le M\,|x-c|^{n+1}.$$

# Example-1 of Taylor Series

**Example Problem 1.2.** Find an approximation for $f(x) = \sin x$, expanded about $c = 0$, using $n = 3$.

*Solution:* Solving for $f^{(k)}$ is fairly easy for this function. We find that

$$\begin{aligned} f(x) = \sin x &= \sin(0) + \frac{\cos(0)\,x}{1!} + \frac{-\sin(0)\,x^2}{2!} + \frac{-\cos(0)\,x^3}{3!} + \frac{\sin(\xi)\,x^4}{4!} \\ &= x - \frac{x^3}{6} + \frac{\sin(\xi)\,x^4}{24}, \end{aligned}$$

so

$$\left|\sin x - \left(x - \frac{x^3}{6}\right)\right| = \left|\frac{\sin(\xi)\,x^4}{24}\right| \le \frac{x^4}{24},$$

because $|\sin(\xi)| \le 1$. ⊣

```
clear, clc
coef = [-1/6,0,1,0]
y = polyval(coef,0.2)
 display('app. value of cos(0.2) is'); y
pause
x = -0.2:.01:0.2
y = polyval(coef,x);
plot(x,y,'r.'); grid; hold on
plot(x,sin(x)); hold off
```

```
% large range
x2=-4:0.01:4;
y2 = polyval(coef,x2);
plot(x2,y2,'r.'); grid
hold on
plot(x2,sin(x2))
hold off
```



# Matlab Functions: polyval(), feval()

Download ployevalexample.m

```
clear, clc

% coeffcients of polynominal:
coef = [-1,1,2]
% function value at point 4
y = polyval(coef,4)
display('value of y = -x^2+x+2 at point x = 4 is')
y
pause
% evaluation of poly. function with an interval
% e.g., y = -x^2+x+2 within [-2,2]
% a vector to represent [-2,2] with step 0.1
x = -2:.1:2
% function values as a vector y
y = polyval(coef,x);
% plotthe function within the defined interval
plot(x,y,'r.'); grid
title('y=-x^2+x+2')
```

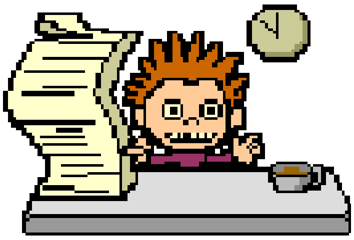
Download erfapp.m

```
% 2 erf function approximation

function y=erfapp(x,p)

y=x.*2/sqrt(pi);
term=1;
for i=1:p
   term=term/(i);
   y=y+ 2/sqrt(pi)*(-1)^i*x.^(2*i+1);
end
```

# Exercises (MM2)





**Question One:**

Regarding to the function $f(x) = cos(x)$,

- Derive the Taylor expansion of it up to 5th order at the point $x = 0$;
- Use the above polynomial to approximate $cos(-0.2)$ and
- Evaluate the approximation error using Taylor's Theorem.

**Question Two:**

(Exercise 3.2.2 and 3.2.2, page 60) Function $ln(1 + x)$ can be approximated by a power series as

$$ln(1+x) = -\sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{k+1}}{k+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots \quad (1)$$

- Write a Matlab m-file to approximate $ln(1.25)$ using the first 6 terms of equation (1);
- How many terms of the series (1) are needed to approximate $ln(1.25)$ with error smaller than $10^{-6}$?
- Use Matlab's built-in function $log()$ to verify that the error of the above second analysis is indeed within the tolerance.

**Question Three:**

The function $erf(x)$ is defined as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \quad (2)$$

This function is often used in the probabilistic analysis of normalized stochastic variable.

- Derive the series approximation of function $e^x$ up to 9th order at point $x = 0$;
- Use the series approximation obtained in last step to approximate $e^{t^2}$ and thereby prove that the series approximation of function $erf$ looks like

$$\widehat{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{9} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}. \quad (3)$$

- Write a Matlab m-file to realize the approximation $\widehat{erf}(x)$ and plot this approximation within interval $[-2, 2]$ and $[0, 4]$, respectively;
- Use the Matlab's built-in function $erf()$ and plot the difference (errors)between this function and approximation $\widehat{erf}(x)$ within interval $[-2, 2]$ and $[0, 4]$, respectively;
- How to evaluate the approximation errors? Use the Matlab function $norm(x, p)$ to evaluate the $L_1$-norm, $L_2$-norm and $L_\infty$-norm of the errors within interval $[-2, 2]$ and $[0, 4]$, respectively.

# MM3 Iterative Solution of Equations

Reading material: Section 2.1, 2.2, 2.3

# Question

- How to represent numbers – MM1
- How to evalue functions – MM2
- How to solve equations?

What's the solution to $x^2+5x+6=0$?

What's the solution to $x^{10}+5x^3+6=0$?

What's the solution to $F(x)=0$?



# Motivation

- $F(x)=0$
- Initial guess $a_0$, check $F(a_0)$? – $error_0$
- Second guess $a_1$, with objective is to try to reduce $error_0$, i.e., $error_1 < error_0$
- Thrid guess $a_2$, with objective is to try to reduce $error_1$, i.e., $error_2 < error_1$
- ….
- $a_n$ has the property: $F(a_n) \to 0$

**Iterative methods for solving equations**

## Concerns Using Iterative Methods

- Sequences
- Convergence of sequences

$$a_n \to L \text{ as } n \to \infty$$

- Computing efficiency
- Iterative methods



## Bisection Method

- Motivation: intermediate value Theorem

  Continuous function f(x) on an interval [a,b] has the property: f(a)f(b)<0, then there is a solution of f(x)=0 between a and b.
- Algorithm
- Example: solution to x-cos(x)=0 with tolerance 0.1
- Matlab codes

| | |
|---|---|
| input | •Equation f(x)=0<br>•Interval [a,b] such that f(a)f(b)<0<br>•Tolerance e |
| repeat | m:=(a+b)/2<br>If f(a)f(m)=< 0, then b:=m else a:=m |
| until | b-a< e |
| output | Solution lies in [a,b] with the length less than e |

# Function Iteration Method

- Motivation:

Rewrite f(x)=0 as x-g(x)=0, i.e.,
x=g(x)
Initial guess x_0
Iterative solution: x_n=g(x_n-1)

How to guarantee the convergence?



# Convergence Theorem for Function Iteration

Suppose that function $g(x)$ is differentiable on $[a,b]$ and that
(1) $g(x) \in [a,b]$ for any $x \in [a,b]$, and
(2) $|\frac{dg(x)}{dx}| \le K < 1$ for all $x \in [a,b]$;

Then, the equation $x = g(x)$ has a unique solution in the interval $[a,b]$ and the iterative sequence defined by

$$x_0 \in [a,b], \quad x_n = g(x_{n-1}), \ n = 1, 2, \cdots$$

converge to this solution

# Example

Find the positive solution of e^x-2x-1=0

- By bisection method
  - Find a and b points such that f(a)f(b)<0 – [1,2]
  - Can you arrange a Matlab m-file to solve this equation with tolerance, e.g., 10e-6?

- By function iteration method
  - Can you determine whether all or some of these formulas are ok using function iteration method? Create your m-file to try!

$$(a) \quad x = \frac{e^x - 1}{2}$$

$$(b) \quad x = e^x - x - 1$$

$$(c) \quad x = \ln(2x+1)$$