MM4 Newton's Method
(Iterative Solution of Equations)

- kl.8.15-9.00 review of MM3 and some examples
- kl.9.00 – 10.30 exercise (see notes)
- kl.10.40-11.30 MM4 lecture (I)

Concerns Using Iterative Methods

- Sequences
- Convergence of sequences

  a_n-> L as n -> ∞
- Iterative methods
  - Bisection method
  - Function iteration method
- Efficient Computing methods
  - Newton's method

Bisection Method (MM3)

- Motivation: intermediate value Theorem

  Continuous function f(x) on an interval [a,b] has the property: f(a)f(b)<0, then there is a solution of f(x)=0 between a and b.

- Algorithm
- Matlab codes

input	•Equation f(x)=0 •Interval [a,b] such that f(a)f(b)<0 •Tolerance e
repeat	m:=(a+b)/2 If f(a)f(m)=< 0, then b:=m else a:=m
until	b-a< e
output	Solution lies in [a,b] with the length less than e

Bisection Method (Matlab function)

```
function sol = bisect(f,a,b,tol)
fa = feval(f,a); fb = feval(f,b);
if fa*fb>0
   fprintf('Two endpoints have same sign\n')
    return
end

tol = max(tol,eps);

while abs(b-a)>tol
    m = (a+b)/2;
    fm = feval(f,m);
    if fa*fm<=0, b=m;
    else a = m;
    end
end
sol = (a+b)/2;
```

Download bisect.m and eq1.m

```
function y = eq1(x)

y = x-cos(x);
```

```
>> format long
>> bisect(@eq1,0,1,10e-5)

ans =

   0.739105224609375
```

Bisection Method - Example

Find the positive solution of e^x-2x-1=0

- By bisection method
  - Find a and b points such that f(a)f(b)<0 – [1,2]
  - Can you arrange a Matlab m-file to solve this equation with tolerance, e.g., 10e-6?

```
function y = eq2(x)

y = exp(x)-2*x-1;
```

```
>> bisect(@eq2,0.1,1,10e-6)
Two endpoints have same sign
>> bisect(@eq2,1,2,10e-6)

ans =

  1.256427764892578
```


Function Iteration Method (MM3)

- Rewrite f(x)=0 as x=g(x)
- Initial guess x_0
- Iterative solution: x_n=g(x_n-1)

Suppose that function $g(x)$ is differentiable on $[a,b]$ and that
(1) $g(x) \in [a,b]$ for any $x \in [a,b]$, and
(2) $|\frac{dg(x)}{dx}| \le K < 1$ for all $x \in [a,b]$;

Then, the equation $x = g(x)$ has a unique solution in the interval $[a,b]$ and the iterative sequence defined by
$x_0 \in [a,b], \; x_n = g(x_{n-1}), \; n = 1, 2, \cdots$
converge to this solution

Function Iteration Method – Example (I)

Find the positive solution of e^x-2x-1=0 by function iteration method. Can you determine whether all or some of these formulas are ok using function iteration method?

$$(a) \quad x = \frac{e^x - 1}{2}$$

$$(b) \quad x = e^x - x - 1$$

$$(c) \quad x = \ln(2x+1)$$

```
clear, clc
format short g
% starting point:
p1 = 1.5; p2 = 1.5; p3 = 1.5;
% iterative solutions
x = zeros(8,3);
% iteration
for k = 1:8
   x(k,:) = [p1,p2,p3];
   p1 = (exp(p1)-1)/2;
   p2 = exp(p2)-p2-1;
   p3 = log(2*p3+1);
end
disp(' #Iterations  eq1 eq2  eq3')
disp([[1:8]' x])
```

#Iterations	eq1	eq2	eq3
1	1.5	1.5	1.5
2	1.7408	1.9817	1.3863
3	2.3511	4.2733	1.3278
4	4.7484	66.485	1.2962
5	57.202	7.4798e+028	1.2788
6	3.4799e+024	Inf	1.2691
7	Inf	NaN	1.2636
8	Inf	NaN	1.2605

Function Iteration Method – Example (II)

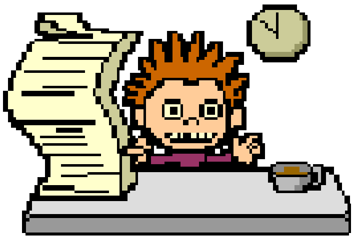
Find the positive solution of e^x-2x-1=0 by function iteration method. Can you determine whether all or some of these formulas are ok using function iteration method?

```
clear, clc
format short g
% starting point:
p1 = 0.5; p2 = 0.5; p3 = 0.5;
% iterative solutions
x = zeros(20,3);
% iteration
for k = 1:20
   x(k,:) = [p1,p2,p3];
   p1 = (exp(p1)-1)/2;
   p2 = exp(p2)-p2-1;
   p3 = log(2*p3+1);
end
disp(' #Iterations  eq1 eq2  eq3')
disp([[1:20]' x])
```

#Iterations	eq1	eq2	eq3
1	0.5	0.5	0.5
2	0.32436	0.14872	0.69315
3	0.19157	0.011628	0.86974
4	0.10558	6.7871e-005	1.0078
5	0.055676	2.3033e-009	1.1038
6	0.028627	0	1.1655
7	0.014521	0	1.2033
8	0.0073132	0	1.2257
9	0.00367	0	1.2388
10	0.0018384	0	1.2463
11	0.00092004	0	1.2507
12	0.00046023	0	1.2531
13	0.00023017	0	1.2546
14	0.0001151	0	1.2554
15	5.7552e-005	0	1.2558
16	2.8777e-005	0	1.2561
17	1.4389e-005	0	1.2562
18	7.1943e-006	0	1.2563
19	3.5972e-006	0	1.2564
20	1.7986e-006	0	1.2564

Why?

Exercises (MM3)



Question One:

Consider the equation

$$3x^3 - 5x^2 - 4x + 4 = 0 \quad (1)$$

- Show that this equation has a root in the interval $[0, 1]$;
- Use the bisection method to obtain an interval of the length less than $1/8$ containing this solution;
- How many iterations would be needed to obtain this solution with an error smaller than 10^{-6}? Write your m-file.
- By using the function iteration method, two rearrangements of equation (1) are carried out as

$$(i) x = 5/3 + 4/(3x) - 4/(3x^2)$$
$$(ii) x = 1 + \frac{3x^3 - 5x^2}{4} \quad (2)$$

  Define your own functions of (i) and (ii) using Matlab m-files and calculate the first 10 iterations for each rearrangement starting with $x_0 = 0.7$;
- Which of the above iterations will converge to a solution near 0.7?
- Find this solution using a tolerance of 10^{-6}.

Question Two:

Consider the equation

$$exp(x) - 100x^2 = 0 \tag{3}$$

- This equation has exactly 3 solutions, obtain the intervals of the length less than 0.1 containing them using bisection method;
- By using the function iteration method, three rearrangements of equation (1) are carried out as

$$\begin{aligned} &(i) x = \frac{exp(x/2)}{10} \\ &(ii) x = 2(lnx + ln10) \\ &(iii) x = \frac{-exp(x/2)}{10} \end{aligned} \tag{4}$$

  Verify that they are all rearrangements of (4);
- Determine which rearrangement will converge to which solution;
- Use these iterations to locate the solutions with tolerance of 10^{-6}.

MM4 Newton's Method

Reading material: section 2.4

Motivation

- Function Iteration Method – Example (II)
- Find the positive solution of e^x-2x-1=0 by function iteration method.

```
clear, clc
format short g
% starting point:
p1 = 0.5; p2 = 0.5; p3 = 0.5;
% iterative solutions
x = zeros(20,3);
% iteration
for k = 1:20
   x(k,:) = [p1,p2,p3];
   p1 = (exp(p1)-1)/2;
   p2 = exp(p2)-p2-1;
   p3 = log(2*p3+1);
end
disp('  #Iterations  eq1 eq2  eq3')
disp([[1:20]' x])
```

#Iterations	eq1	eq2	eq3
1	0.5	0.5	0.5
2	0.32436	0.14872	0.69315
3	0.19157	0.011628	0.86974
4	0.10558	6.7871e-005	1.0078
5	0.055676	2.3033e-009	1.1038
6	0.028627	0	1.1655
7	0.014521	0	1.2033
8	0.0073132	0	1.2257
9	0.00367	0	1.2388
10	0.0018384	0	1.2463
11	0.000[illegible]	[illegible]	[illegible]
12	0.000[illegible]	[illegible]	[illegible]
13	0.000[illegible]	[illegible]	[illegible]
14	0.000[illegible]	[illegible]	[illegible]
15	5.755[illegible]	[illegible]	[illegible]
16	2.8777e-005	0	1.2561
17	1.4389e-005	0	1.2562
18	7.1943e-006	0	1.2563
19	3.5972e-006	0	1.2564
20	1.7986e-006	0	1.2564

Both equation 1 and 2 converge to Solution 0, but equation 2 leads Faster convergence

Iteration Method – Converging Rate

assume the solution to $x = g(x)$ is s, and an iterative solution is x_n,
then the sequence of errors in the iterates is

$$e_n = x_n - s$$

$$e_{n+1} = x_{n+1} - s = g(x_n) - s = g(s) + g'(s)(x_n - s) + \frac{g''(s)}{2!}(x_n - s)^2 + \cdots - g(s)$$

$$e_{n+1} = g'(s)(x_n - s) + \frac{g''(s)}{2!}(x_n - s)^2 + \cdots \approx g'(s)(x_n - s) = g'(s)e_n$$

Concern the example: s=0

$$(i) \quad g(x) = \frac{e^x - 1}{2}, \quad g'(x) = \frac{e^x}{2}, \quad e_{n+1} \approx \frac{1}{2} e_n$$

$$(ii) \quad g(x) = e^x - x - 1, \quad g'(x) = e^x - 1, \qquad g''(x) = e^x, \quad e_{n+1} \approx \frac{1}{2!} {e_n}^2$$

quadratic (second-order convergence)

Newton's Method

- Newton-Raphson method

Suppose to solve $f(x)=0$

The first-order Taylor expansion of f about a point x_0 is

$$f(x) \approx f(x_0)+(x-x_0)f'(x_0)$$

If the point x_0 is close to the required solution, then we expect that

$$f(x_0)+(x-x_0)f'(x_0)=0$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n=0,1,2,\cdots$$

$$g(x) = x - \frac{f(x)}{f'(x)}$$

$$g'(x) = 1 - \frac{f'(x)}{f'(x)} + \frac{f(x)f''(x)}{(f'(x))^2}$$

$$=1-1+0=0$$

Convergence Theroem

Let f be twice differentiable on an interval $[a,b]$ and satisfy the conditions

(*i*) $f(a)f(b)<0$

(*ii*) $f'(x)$ has no zeros on $[a,b]$

(*iii*) $f''(x)$ does not change sign in $[a,b]$

(*iv*) $\left|\frac{f(a)}{f'(a)}\right|, \left|\frac{f(b)}{f'(b)}\right| < b-a$

Then f(x)=0 has a unique solution and Newton's iteration will converge to this solution from any starting point in $[a,b]$