System Expressions:

- Impulse Response sequence $h[n]$ $\begin{cases} \text{IIR} \\ \text{FIR} \end{cases}$
- System function $H(z)$ pole-zero diagram, stable ...
- Frequency Response $H(e^{j\omega})$ $\begin{cases} \text{Magnitude} \\ \text{phase} \end{cases}$, linear phase ..
- Difference equation $\underbrace{\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]}_{\text{Structure}}$

Example 1.

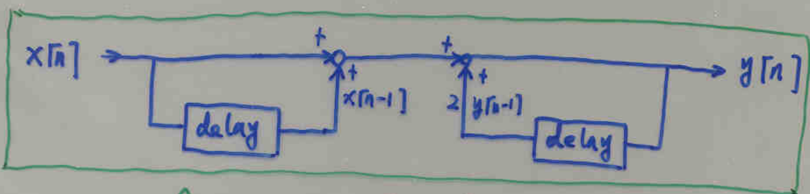
$$H(z) = \frac{1+z^{-1}}{1-2z^{-1}} = \frac{1}{1-2z^{-1}} + \frac{z^{-1}}{1-2z^{-1}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-1}}{1-2z^{-1}}$$

$$\mathcal{Z}^{-1}\{(1-2z^{-1})Y(z)\} = \mathcal{Z}^{-1}\{(1+z^{-1})X(z)\}$$

$$y[n] - 2y[n-1] = x[n] + x[n-1]$$

$$y[n] = 2y[n-1] + x[n] + x[n-1]$$

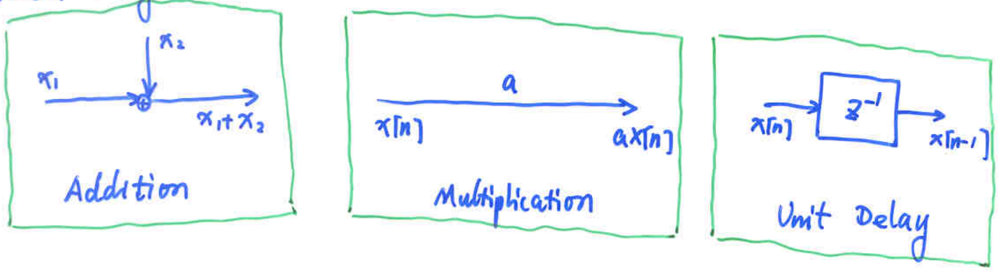


Recursive Computation

$$h[n] = 2^n u[n] + 2^{n-1} u[n-1] \leftarrow \text{IIR!}$$

$$y[n] = h[n] * x[n] = \sum_{k=0}^{\infty} h[k] x[n-k]$$

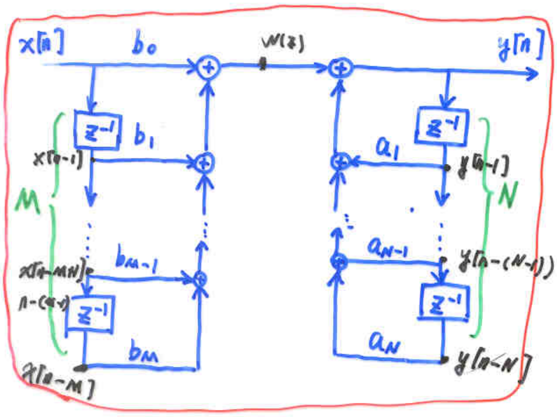
Block Diagram

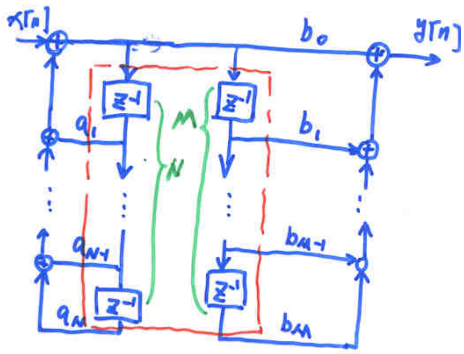


General Case:

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$





$$H(z) = \underbrace{\left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right)}_{\text{poles part}} \underbrace{\left(\sum_{k=0}^{M} b_k z^{-k} \right)}_{\text{zeros part}}$$

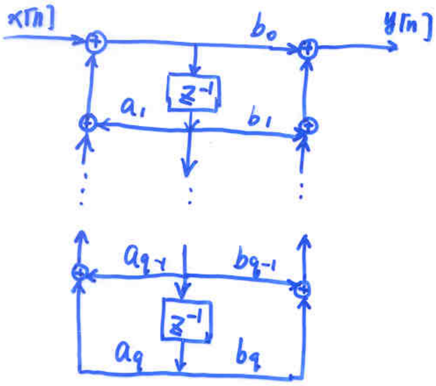
$$H(z) = \left(\sum_{k=0}^{M} b_k z^{-k} \right) \left(\frac{1}{1 - \sum_{k=1}^{N} a_k z^{-k}} \right)$$

* Each arrangement represents a different computation algorithm for implementing the same system.

* When different arrangement is implemented with finite precision arithmetic, there can be a significant difference !

Canonic form implementation:

An implementation with the minimum number of delays.



* Direct form I implementation

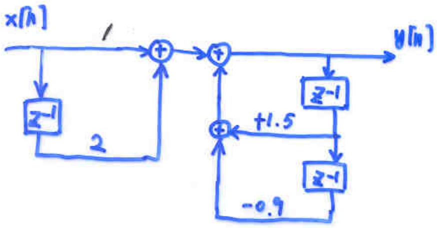
* Direct form II implementation (Canonical direct form)

$q = \max\{M, N\}$

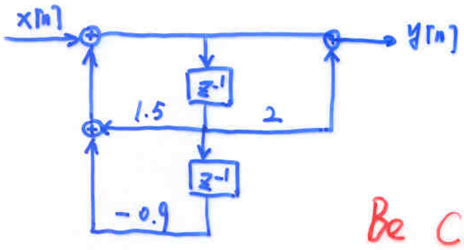
Example 2:

$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}$$

$$1 - (1.5z^{-1} - 0.9z^{-2})$$



Direct form I implementation



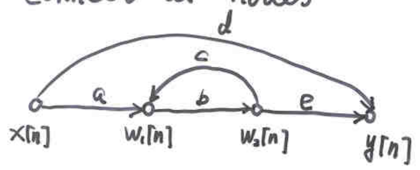
Direct form II implementation

Be Careful of $\{a_k\}$ signs!

There are unlimited number of equivalent realizations of any given system (based on the linearity and algebraic properties

Signal Flow Graph

* A signal flow graph is a network of directed branches that connect at nodes



directed branches (j, k)

nodes $w_i[n]$

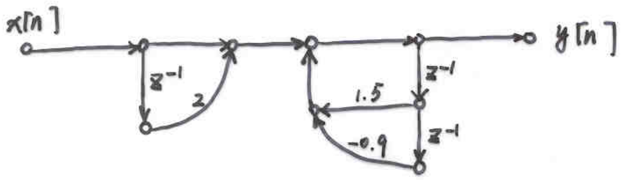
- source nodes
- sink nodes

$$w_1[n] = a x[n] + c w_2[n]$$
$$w_2[n] = b w_1[n]$$
$$y[n] = d x[n] + e w_2[n]$$

$$y[n] = \left(\frac{abe}{1-bc} + d\right) x[n]$$

Example 2 continue ...

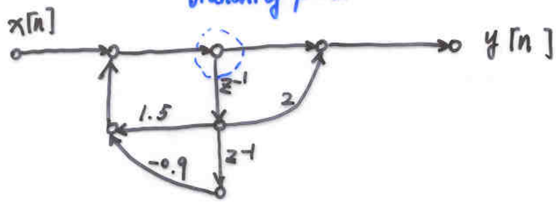
(a)



signal flow graph of the direct I form

branching point

(b)

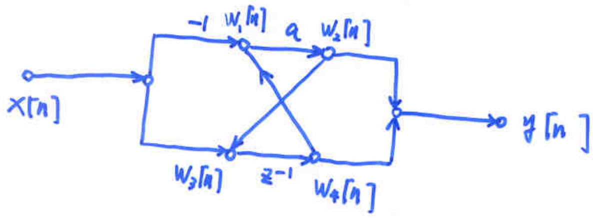


signal flow graph of the direct II form.

A signal flow graph theory exists that can be directly applied to discrete-time systems when they are represented in this form

Determine the system function from signal flow graph

Example 3 :



$$W_1[n] = W_4[n] - x[n]$$

$$W_2[n] = a\,W_1[n]$$

$$W_3[n] = x[n] + W_2[n]$$

$$W_4[n] = W_3[n-1]$$

$$y[n] = W_2[n] + W_4[n]$$

$\Rightarrow$

$$W_1(z) = W_4(z) - X(z)$$

$$W_2(z) = a\,W_1(z)$$

$$W_3(z) = X(z) + W_2(z)$$

$$W_4(z) = z^{-1}\,W_3(z)$$

$$Y(z) = W_2[n] + W_4[n]$$

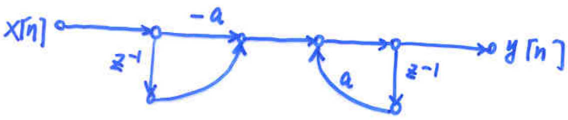
$\Downarrow$

$$Y(z) = \frac{z^{-1} - a}{1 - a z^{-1}}\;X(z)$$

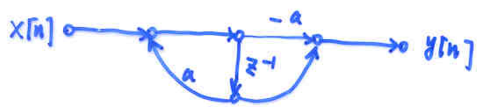
$$H(z) = \frac{z^{-1} - a}{1 - a z^{-1}}$$

$$h[n] = a^{n-1}u[n-1] - a^{n+1}u[n]$$

(a) Form I :



(b) Form II :

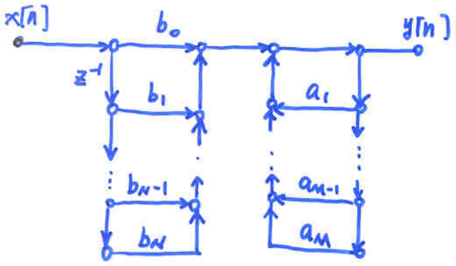


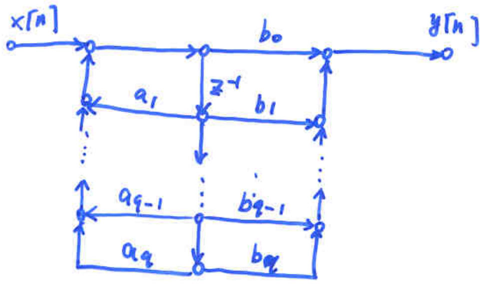
Computational Complexity $\begin{cases} \text{Multiplication : time-consuming and costly operation} \\ \text{Delay : memory registers} \end{cases}$

Basic Structures For IIR Systems

* Direct forms $\begin{cases} \text{Direct Form I} \\ \text{Direct Form II} \end{cases}$

General Case :
$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k u[n-k]$$





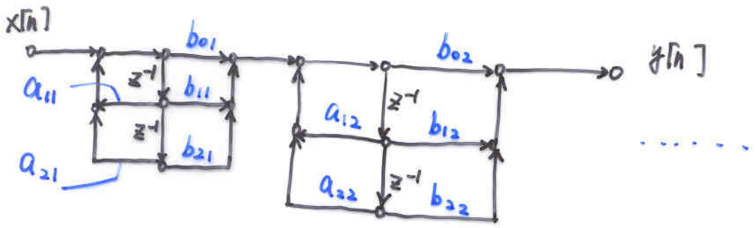
$q = \max\{M, N\}$

* Cascade Form

$$H(z) = A \frac{\overbrace{\prod_{k=1}^{M_1}(1-f_k z^{-1})}^{\text{real zeros}} \overbrace{\prod_{k=1}^{M_2}(1-g_k z^{-1})(1-g_k^* z^{-1})}^{\text{conjugate pair of zeros}}}{\underbrace{\prod_{k=1}^{N_1}(1-c_k z^{-1})}_{\text{real poles}} \underbrace{\prod_{k=1}^{N_2}(1-d_k z^{-1})(1-d_k^* z^{-1})}_{\text{conjugate pair of poles}}}$$

$\Downarrow$

$$H(z) = \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$



$\begin{cases} * \ N_s! \text{ pairings of the poles and zeros} \\ * \ N_s! \text{ orderings of the resulting second-order sections} \end{cases}$

Remark about the cascade form

- Five-multiplier second-order module

$$H(z) = \prod_{k=1}^{N_s} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}} \quad \text{(Fixed-point arithmetic)}$$

- Four-multiplier second-order module

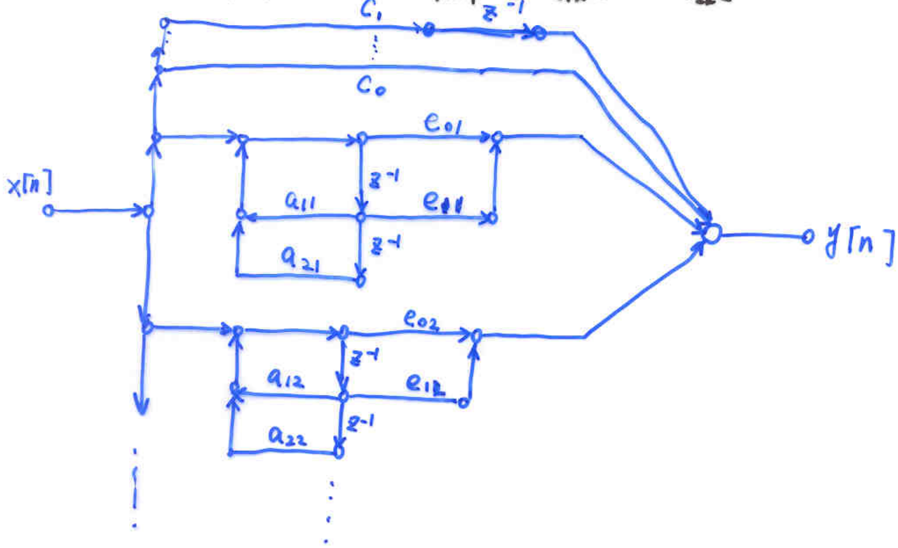
$$H(z) = b_0 \prod_{k=1}^{N_s} \frac{1 + \tilde{b}_{1k} z^{-1} + \tilde{b}_{2k} z^{-2}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}} \quad \text{(floating-point arithmetic)}$$

* Parrallel Form

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_1} \frac{A_k}{1 - C_k z^{-1}} + \sum_{k=1}^{N_2} \frac{B_k (1 - e_k z^{-1})}{(1 - d_k z^{-1})(1 - d_k^* z^{-1})}$$

$\Downarrow$ A parrallel combination of first- and second-order IIR systems

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_s} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

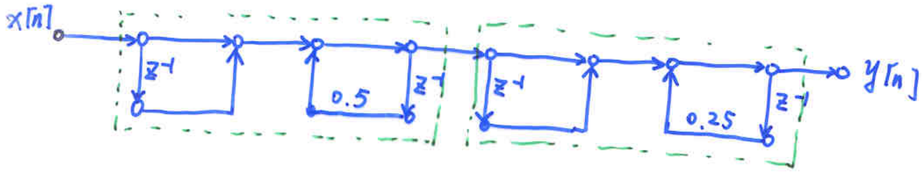


Example 3: $H(z) = \dfrac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$

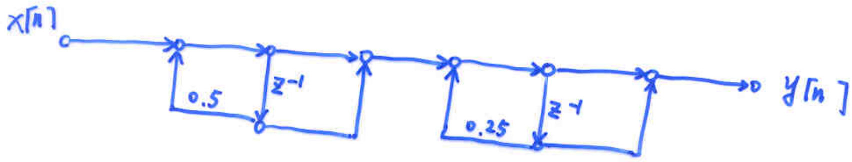
(a) Cascade-Form Structures

$$H(z) = \frac{(1+z^{-1})(1+z^{-1})}{(1-0.5z^{-1})(1-0.25z^{-1})} = \frac{1+z^{-1}}{1-0.5z^{-1}} \cdot \frac{1+z^{-1}}{1-0.25z^{-1}}$$

Direct form I



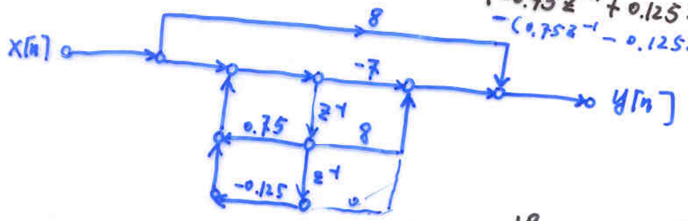
Direct form II



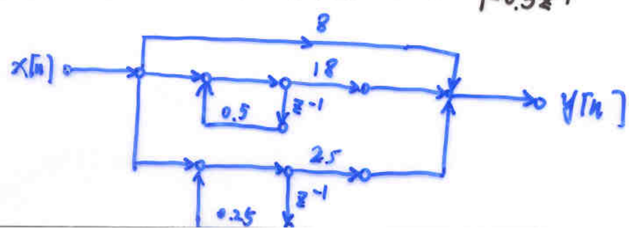
(b) Parrallel Form Structures

- second-order-Form $H(z) = 8 + \dfrac{-7 + 8z^{-1} + 0.z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$

$-(0.75z^{-1} - 0.125z^{-2})$



- First-order-Form $H(z) = 8 + \dfrac{18}{1-0.5z^{-1}} - \dfrac{25}{1-0.25z^{-1}}$



Remarks :

* For a network with no loops, the impulse response is finite and no longer than the total number of delay elements
* Neither poles in the system function nor loops in the network are sufficient for the impulse response to be infinite.

  e.g., $H(z) = \dfrac{1-a^2z^{-2}}{1-az^{-1}}$
* All loops must contain at least one unit delay element so as to be computable.

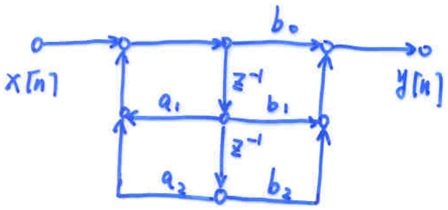
* Transposed forms

Transposition of a flow graph:

(i) Reversing the directions of all branches

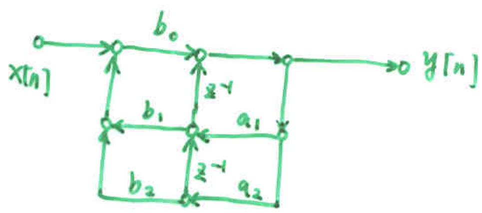
(ii) keeping the branch transmittances

(iii) Reversing the input and output



Direct form II

$\dfrac{Y(z)}{X(z)} = H_1(z)$

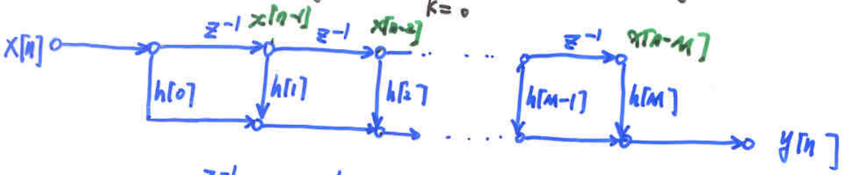


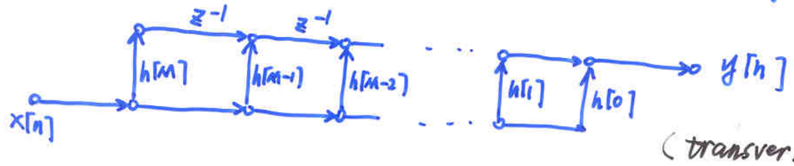
Transposed direct form II

$\dfrac{Y(z)}{X(z)} = H_2(z)$

Basic Structures for FIR Systems

* Direct Form $\quad y[n] = \sum_{k=0}^{M} h[k]\, x[n-k]$

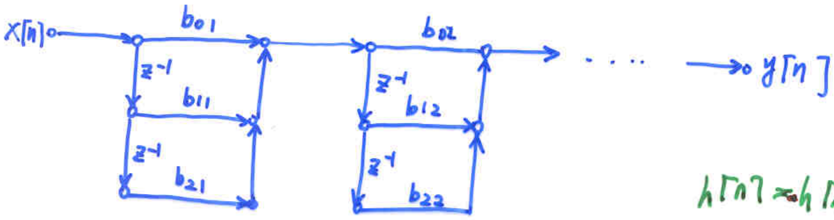




(transversal filter structure

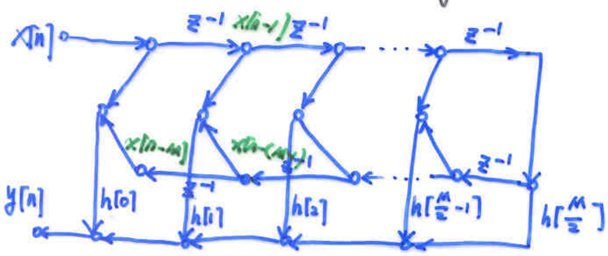
* Cascade Form

$$H(z) = \sum_{n=0}^{M} h[n]\, z^{-n} = \prod_{k=1}^{M_s} (b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2})$$



$h[n] = h[M-n]$
M is even

* Linear Phase FIR systems



when M is even

(type I or III)

$h[0] = h[M]$

$y[n] = \sum_{k=0}^{M} h[k]\, x[n-k]$

when M is odd, see P369. Fig 6.35

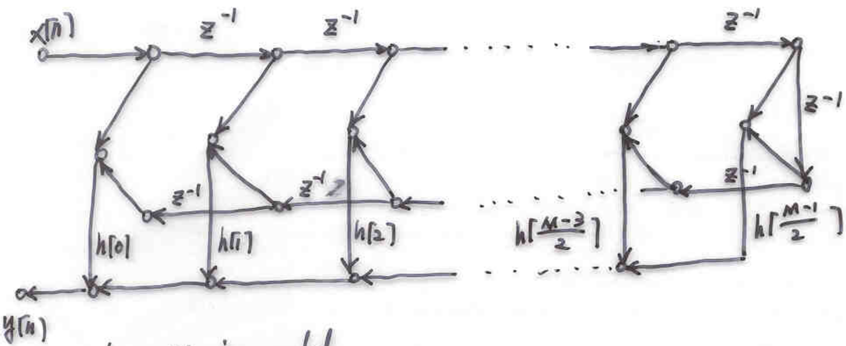
$= h[0]x[n] + h[M]x[n-M]$

$h[0] \cdot (x[n] + x[n-M])$

Remarks:

* One motivation for considering alternatives to the simple direct form structures is that different structures that are theoretically equivalent may behave differently when implemented with finite numerical precision.

* The implementation structure determines the quantization noise generated internally in the system



when M is odd

$$y[n] = \sum_{k=0}^{\frac{M-1}{2}} h[k]\left(x[n-k] + x[n-M+k]\right) \quad (\text{type II})$$

$$y[n] = \sum_{k=0}^{\frac{M-1}{2}} h[k]\left(x[n-k] - x[n-M+k]\right) \quad (\text{type IV})$$