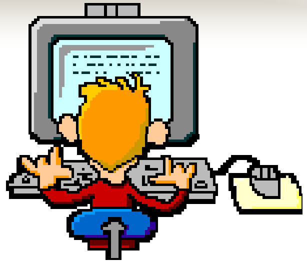
Reading material: **p.160-163, 439 – 458 and 824-829**

MM2: Synthesis of IIR DT filters

1. Explanation of last exercise
2. Continuous time filters
3. Impulse-invariance method
4. Bilinear transformation method

Explanation of Exercise One

Analog filter



Discrete filter

Effect of Filtering

- System frequency response:

$$H(e^{j\omega}) = |H(e^{j\omega})|\, e^{\angle H(e^{j\omega})}$$

- Input and output relationship

$$|Y(e^{j\omega})| = |H(e^{j\omega})|\, |X(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Requirements for Filter Design

- **Frequency-selective filters**

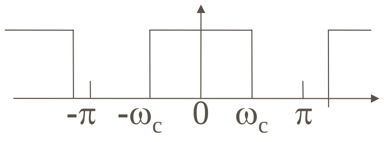
  Lowpass, highpass, bandpass, bandstop filters…

- **Linear Phase filters**
- **Causal filters**
- **Stable filters**

Frequency-Selective Filters

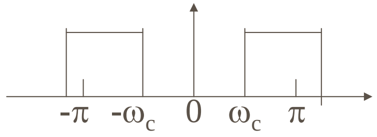
- Ideal lowpass filter

$$|H_{lp}(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega c \\ 0, & others \end{cases}$$



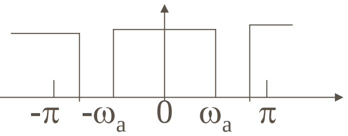
- Ideal highpass filter

$$|H_{hp}(e^{j\omega})| = \begin{cases} 1, & |\omega| \ge \omega c \\ 0, & others \end{cases}$$



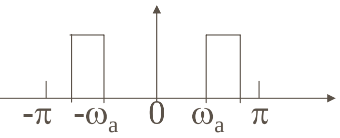
- Ideal bandstop filter

$$|H_{bs}(e^{j\omega})| = \begin{cases} 1, & |\omega| \le \omega_a \text{ and } |\omega| \ge \omega_{ba} \\ 0, & \omega_a \le |\omega| \le b_a \end{cases}$$



- Ideal bandpass filter

$$|H_{bp}(e^{j\omega})| = \begin{cases} 0, & |\omega| \le \omega_a \text{ and } |\omega| \ge \omega_{ba} \\ 1, & \omega_a \le |\omega| \le b_a \end{cases}$$



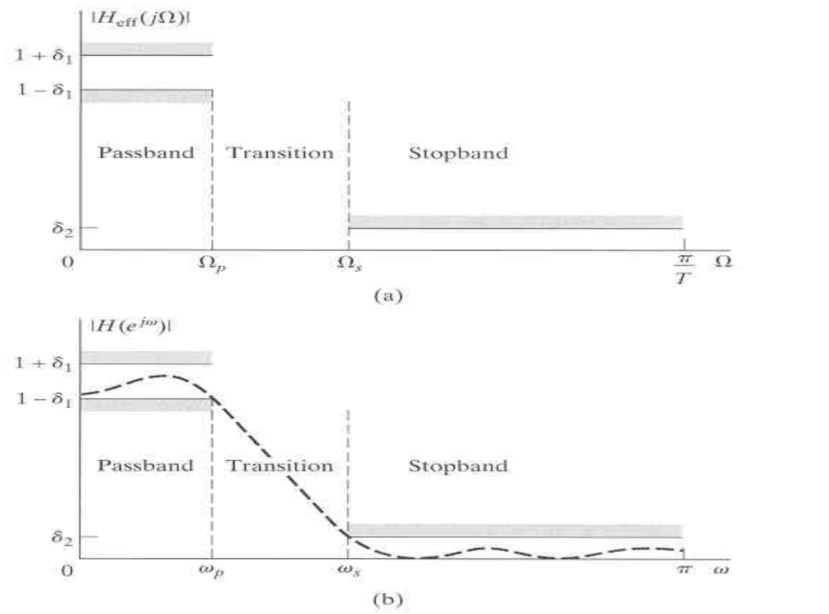


Figure 7.2 (a) Specifications for effective frequency response of overall system in Figure 7.1 for the case of a lowpass filter. (b) Corresponding specifications for the discrete-time system in Figure 7.1.

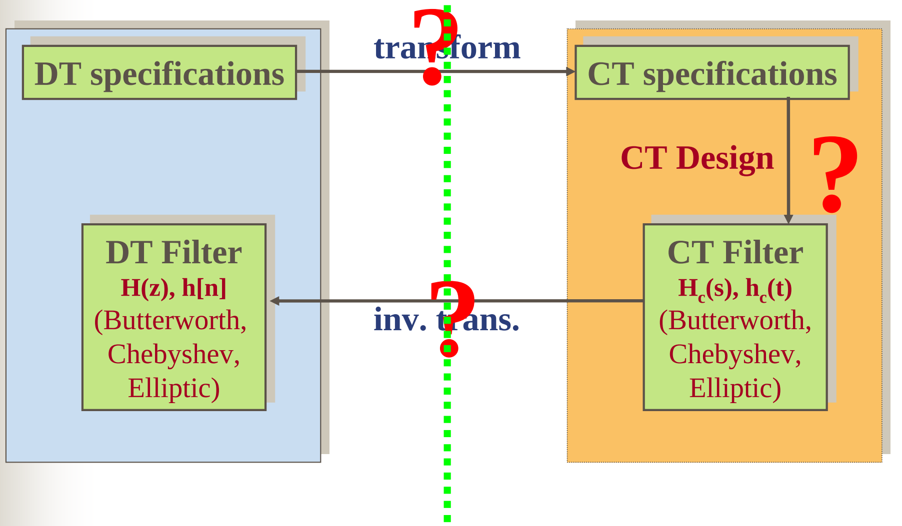
Exercise MM2

3. Design a lowpass DT filter using the impulse-invariance method, such that
   - maximal flat character in passband;
   - DC gain: 0 dB
   - Gain at 750Hz: minimum -1.0dB
   - Gain at 1500Hz: maximum -10.0dB
   - Sampling frequency: 8000Hz
4. Design a lowpass DT filter using the bilinear transformation, which l
   exercise.

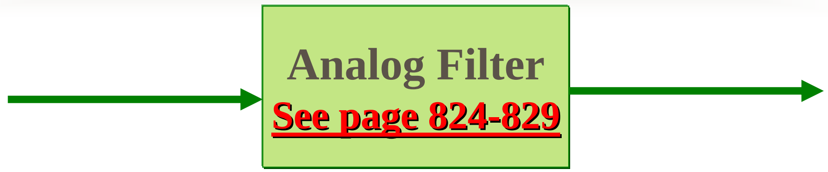
Synthesis of Discrete-Time IIR Filters from Continuous-Time Filters

Discrete-time domain...
Sequences, Z-transform, Fourier Tran...

Continuous-time domain...
functions, Laplace-transform, Fourier ...



Synthesis of Continuous-Time Filters



Butterworth Lowpass Filters (I)

- **Characteristics**
  - The magnitude response is maximally flat in the passband
  - The magnitude response is monotonic in the passband and stopband
- **Magnitude-squared function**

$$|H_c(j\Omega)|^2 = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

$$H_c(j\Omega)H_c(-j\Omega) = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

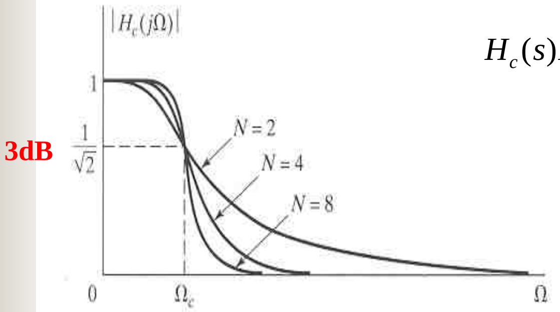
Butterworth Lowpass Filters (II)

- **Filter Construction**
  - System function

$$H_c(s)H_c(-s) = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

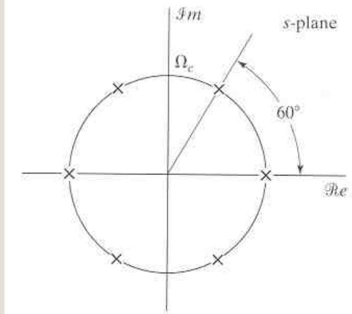
  - Roots of denominator polynomial see Fig.8.3 (p.826)
  - Stable and causal system $H_c(s)$

Select the poles on the left-half-plane of the s-plane



$$H_c(s)H_c(-s) = \frac{1}{1+(j\Omega/j\Omega_c)^{2N}}$$

Figure B.2 Dependence of Butterworth magnitude characteristics on the order *N*.



Chebyshev Filters (I)

- **Motivation:** to distribute the accuracy of the approximation over the passband or the stopband, leading to a lower order filter
- **Characteristics**
  - **Type I Chebyshev filter:** the magnitude response is equiripple in the passband and monotonic in the stopband
  - Type II Chebyshev filter: the magnitude response is monotonic in the passband and equiripple in the stopband
- **Magnitude-squared function of type I**

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 V_N^{\,2}(\Omega/\Omega_c)} \qquad V_N(x) = \cos(N\cos^{-1}x)$$

Where $V_N(x)$ is the Nth-order Chebyshev polynomial, which can be recurrently calculated by

$$V_{N+1}(x) = 2xV_N(x) - V_{N-1}(x)$$

continue..

Chebyshev Filters (II)

- **Design parameters**
  - **ε** can be specified by the allowable passband ripple
  - **Ω_c** can be specified by the desired cutoff frequency
  - **N** can be chosen that the stopband specification are met
- **Location of poles:** on an ellipse
  - Length of the minor axis – **$2a\Omega_c$**
  - Length of the major axis – **$2b\Omega_c$**

$$a = \frac{1}{2}(\alpha^{1/N} - \alpha^{-1/N})$$

$$b = \frac{1}{2}(\alpha^{1/N} + \alpha^{-1/N})$$

$$\alpha = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$$

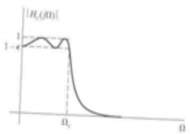


Figure 8.4 Type I Chebyshev lowpass filter approximation.

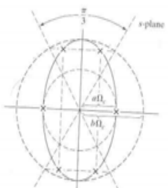


Figure 8.5 Location of poles for a third-order type I lowpass Chebyshev filter.

- **Type II chebyshev filters**

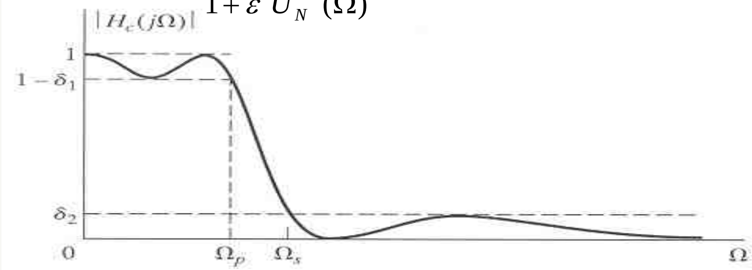
$$|H_c(j\Omega)|^2 = \frac{1}{1+[\varepsilon^2 V_N^{\,2}(\Omega_c/\Omega)]^{-1}}$$

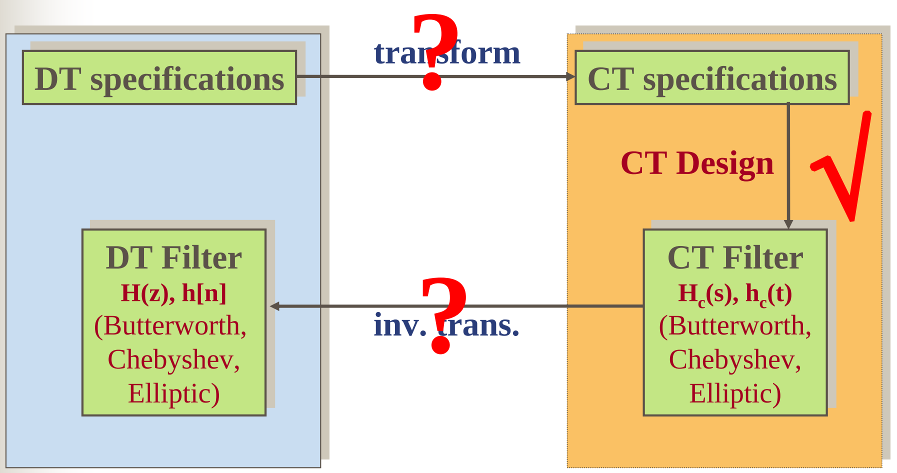
Elliptic Filters

- **Characteristics:** equiripple in the passband and stopband
- Elliptic filters is the best that can be achieved for a given filter order N, in the sense that for a given Ω_p, δ_1, δ_2, the transition band (Ω_s - Ω_p) is as small as possible
- **Magnitude-squared function**

$$|H_c(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N^{\,2}(\Omega)}$$

$U_N(\Omega)$ is a Jacobian elliptic function





Synthesis of Discrete-Time IIR Filters using Impulse Invariance Method

Impulse Invariance Method

- Fundamental formula:

$$h[n] = T_d h_c(nT_d),$$

- Frequency property:

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T_d}) \quad |\omega| \le \pi$$

- DT specification → CT specification: $\Omega=\omega/T_d$
- Pole relationship: (p.445) (stable CT filter →stable DT filter)

$$z_k = e^{s_k T_d}$$

Practical Procedure

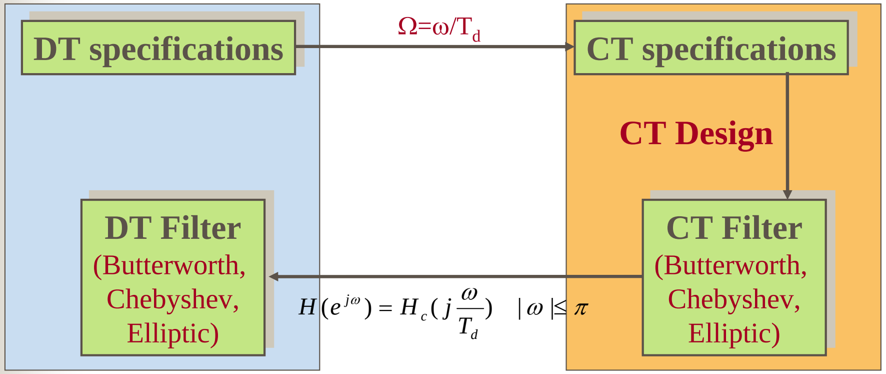
- **Step1:** Transform DT specification to CT specification using

$$\Omega=\omega/T_d$$

- **Step2:** Design a CT filter based on CT specification
- **Step3:** Transform $H_c(s)$ to be $H(z)$

$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \quad \Rightarrow \quad H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

DT IIR Filter by Impulse Invariance



$$H_c(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k} \quad \Rightarrow \quad H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$$

Example: Impulse Invariance Design (p.446)

- Desire a lowpass DT filter with

$$0.89125 \le |H(e^{j\omega})| \le 1, \quad 0 \le |\omega| \le 0.2\pi$$
$$|H(e^{j\omega})| \le 0.17783, \quad 0.3\pi \le |\omega| \le \pi$$

- Select $T_d=1$, CT specification

$$0.89125 \le |H_c(j\Omega)| \le 1, \quad 0 \le |\Omega| \le 0.2\pi$$
$$|H(j\Omega)| \le 0.17783, \quad 0.3\pi \le |\Omega| \le \pi$$

- CT butterworth filter

$$0.89125 \le |H_c(j0.2\pi)|, \quad |H(j0.3\pi)| \le 0.17783$$
$$|H_c(j\Omega)|^2 = \frac{1}{1 + (\Omega / \Omega_c)^{2N}}$$

- Sixth-order $H_c(s)$
- **Matlab auto-design …**

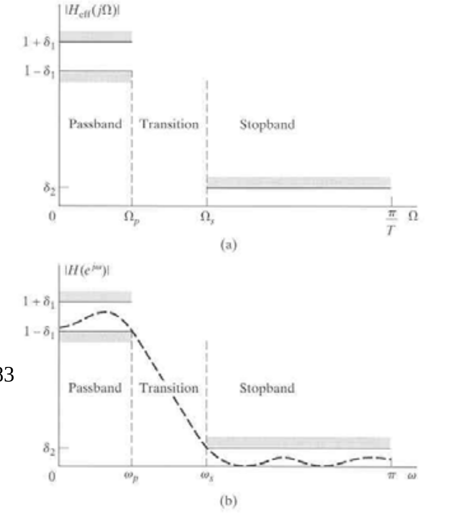


Figure 7.2 (a) Specifications for effective frequency response of overall system in Figure 7.1 for the case of a lowpass filter. (b) Corresponding specifications for the discrete-time system in Figure 7.1.

Remarks about Impulse Invariance

- The basis for this method is to choose a DT impulse response that is similar to the CT impulse response

$$h[n] = T_d h_c(nT_d),$$

- The CT and DT frequencies have linear relationship, except for aliasing, the shape is preserved

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T_d}) \quad |\omega| \le \pi$$

- This method is appropriate only for bandlimited filters

$$H_c(j\Omega)=0, \quad \pi/T_d \le |\Omega|,$$

Synthesis of Discrete-Time IIR Filters using Bilinear Method

Bilinear Transform

- Fundemantal formula

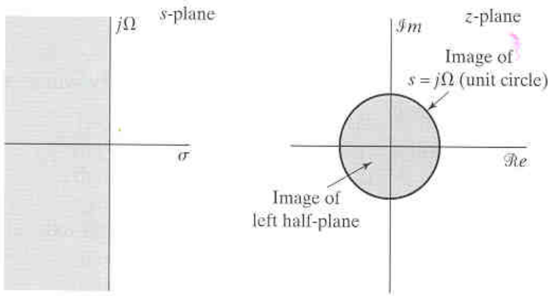
$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right), \quad H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right)$$

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}, \quad H_c(s) = H\left(\frac{1 + (T_d/2)s}{1 - (T_d/2)s}\right)$$

Characteristics of Bilinear Transform

- An algebraic transformation between s-plane and z-plane, i.e., mapping the $j\Omega$-axis in s-plane to one revolution of the unit circle in the z-plane
- the CT and DT frequencies have nonlinear relationship

$$\Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \qquad \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$$



$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

$$z = \frac{1 + (T_d/2)s}{1 - (T_d/2)s}$$

Figure 7.6 Mapping of the *s*-plane onto the *z*-plane using the bilinear transformation.

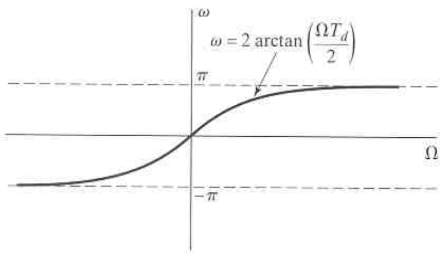
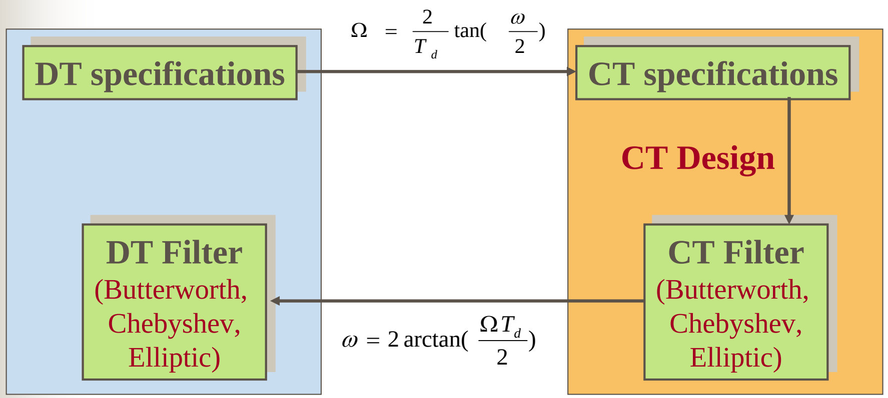


Figure 7.7 Mapping of the continuous-time frequency axis onto the discrete-time frequency axis by bilinear transformation.

DT IIR Filter Design by Bilinear Transform



$$s = \frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right), \quad H(z) = H_c\left(\frac{2}{T_d}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)\right)$$

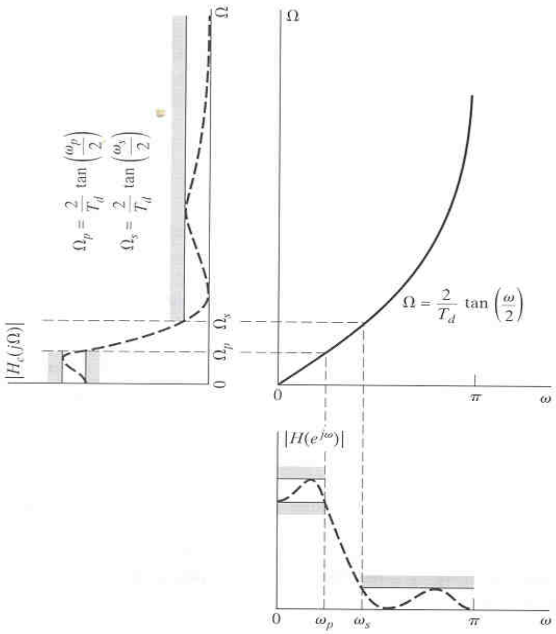


Figure 7.8 Frequency warping inherent in the bilinear transformation of a continuous-time lowpass filter into a discrete-time lowpass filter. To achieve the desired discrete-time cutoff frequencies, the continuous-time cutoff frequencies must be prewarped as indicated.

Remarks about Bilinear Method

- **Sampling period** T_d will not affect the design, therefore, in specific problems it can be chosen as any convenient value
- Stable CT filter → stable DT filter
- Avoid the aliasing problem
- the CT and DT frequencies have nonlinear relationship

$$\begin{cases} \Omega = \frac{2}{T_d}\tan\left(\frac{\omega}{2}\right) \\ \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right) \end{cases}$$

Linear phase property will lose !!

$$e^{-j\Omega\alpha} = e^{-j\frac{2}{T_d}\tan\left(\frac{\omega}{2}\right)}$$

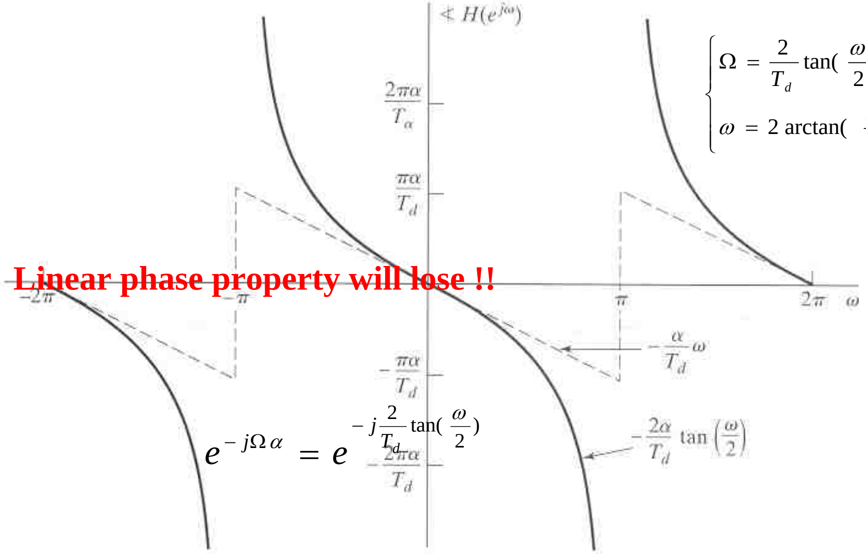


Figure 7.9 Illustration of the effect of the bilinear transformation on a linear phase characteristic. (Dashed line is linear phase and solid line is phase resulting from bilinear transformation.)

Examples of bilinear transform design (see p.454-465)