SE Course: Digital Filter Design

Exercise Solution for MM2

1. Calculate the *DC-gain* and the *cutoff frequency* of the analog lowpass filter $H(s) = 1/(1 + 0.001s)$

   DC-gain: $20log_{10}(|H(0)|) = 20log_{10}(1) = 0dB$
   Cutoff frequency Ω_c:

$$
\begin{aligned}
& -3dB = 20log_{10}(1/\sqrt{2}) = 20log_{10}(|1/(1 + j0.001\Omega_c)|) \\
\Rightarrow \quad & \sqrt{1^2 + 10^{-6}\Omega_c^2} = \sqrt{2} \\
\Rightarrow \quad & \Omega_c = 1000 \; rad/sec \text{ or } \Omega_c = 1000/2\pi \approx 159Hz
\end{aligned}
$$

2. Calculate the *DC-gain* and the *cutoff frequency* of the obtained discrete-time lowpass filter $H(e^{j\omega}) = 0.125/(1 - 0.8825e^{-j\omega})$

   DC-gain: $20log_{10}(|H(e^{j0})|) = 20log_{10}(0.125/(1 - 0.8825)) = 0.5372dB$
   Cutoff frequency Ω_c:

$$
\begin{aligned}
& 0.5372 - 3dB = 20log_{10}(|H(e^{j\omega_c})|) = 20log_{10}(|0.125/(1 - 0.8825e^{-j\omega_c}|)) \\
\Rightarrow \quad & 0.7532 = |0.125e^{j\omega_c}/(e^{j\omega_c} - 0.8825)| \\
\Rightarrow \quad & 0.7532 = 0.125/\sqrt{(cos\omega_c - 0.8825)^2 + (sin\omega_c)^2} \\
\Rightarrow \quad & \omega_c = 0.125 \; rad/sample \text{ or} \\
& \omega_c = 0.125 * 8000/(2\pi) \approx 159Hz
\end{aligned}
$$

3. Design a lowpass DT filter using the impulse-invariance method, such that

   - maximal flat character in passband;
   - DC gain: 0 dB
   - Gain at 750Hz: minimum -1.0dB
   - Gain at 1500Hz: maximum -10.0dB
   - Sampling frequency: 8000Hz

   Step 1: Obtain the specifications for the wanted DT filter.

   - maximal flat character in passband $\Rightarrow$ Butterworth filter
   - DC gain: 0 dB $\Rightarrow |H(e^{j0})| = 1$;
   - Gain at 750Hz: minimum -1.0dB $\Rightarrow$ the passband frequency $\omega_p = 2\pi * 750/8000 = 3\pi/16$ rad/sample, and the system magnitude at this frequency point is $20log_{10}\delta_1 = -1dB \Rightarrow \delta_1 = 0.89125$.
   - Gain at 1500Hz: maximum -10.0dB $\Rightarrow$
     the stopband frequency is $\omega_s = 2\pi * 1500/8000 = 3\pi/8$ rad/sample, and the system magnitude at this frequency point is $20log_{10}\delta_2 = -10dB \Rightarrow \delta_2 = 0.31623$.

   Step 2: Obtain the specifications for the wanted CT filter.

   - Butterworth filter
   - DC gain: 0 dB $\Rightarrow |H_c(0)| = 1$;
   - Using the transform $\Omega = \omega/T_d$:

$$
\begin{aligned}
& 0.89125 \le |H_c(j\Omega)| \le 1, \quad 0 \le |\Omega| \le (\Omega_p = 3\pi/16 * 8000 = 1500\pi) \\
& 0 \le |H_c(j\Omega)| \le 0.31623, \quad (\Omega_s = 3\pi/8 * 8000 = 3000\pi) \le |\Omega| \le \pi
\end{aligned}
$$

   Step 3: Design a CT Butterworth filter which satisfies above specifications using $|H_c(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$. With respect to the monotonic magnitude property of Butterworth filter, there is

$$
\frac{1}{1 + (1500\pi/\Omega_c)^{2N}} = 0.89125^2
$$

$$
\frac{1}{1 + (3000\pi/\Omega_c)^{2N}} = 0.31623^2
$$

From above equations, we can obtain that $N = 2.5792$. Because N should be some integer, we take $N = 3$. Then, from the first equation there is $\Omega_c = 5902.7$ (1878.9π) rad/sec. From the second equation there is $\Omega_c = 6534.8$ (2080.1π) rad/sec. In order to let both requirements for passband and stopband satified, the parameter Ω_c should satisfies $5902.7 \leq \Omega_c \leq 6534.8$.

In the following, we take $\Omega_c = 5902.7$ and $N = 3$. Then, the 6 poles of the magnitude-squared function $H_c(s)H_c(-s) = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$ are uniformly distributed in the angle on a circle of radius $\Omega_c = 5902.7$. Then, the poles of $H_c(s)$ are the three ones in the left half of the s-plane, i.e., $-\Omega_c, -\Omega_c/2 \pm j\sqrt{3}\Omega_c/2$. Therefore, there is

$$H_c(s) = \frac{K}{(s+\Omega_c)(s^2+\Omega_c s+\Omega_c^2)}.$$

Considering the DC-gain is 0 dB, it means that $K = \Omega_c^3$.

Step 4: Obtain the DT lowpass filter as

$$H_c(s) = \frac{\Omega_c^3}{(s+\Omega_c)(s^2+\Omega_c s+\Omega_c^2)}.$$

4. Design a lowpass DT filter using the bilinear transformation, which has the same specifications as above exercise.

   see the second part of solutions

5. Begin to be familiar with Filter Design and Analysis Tool in Matlab

   Open Matlab, then type

   $\gg$ fdatool...