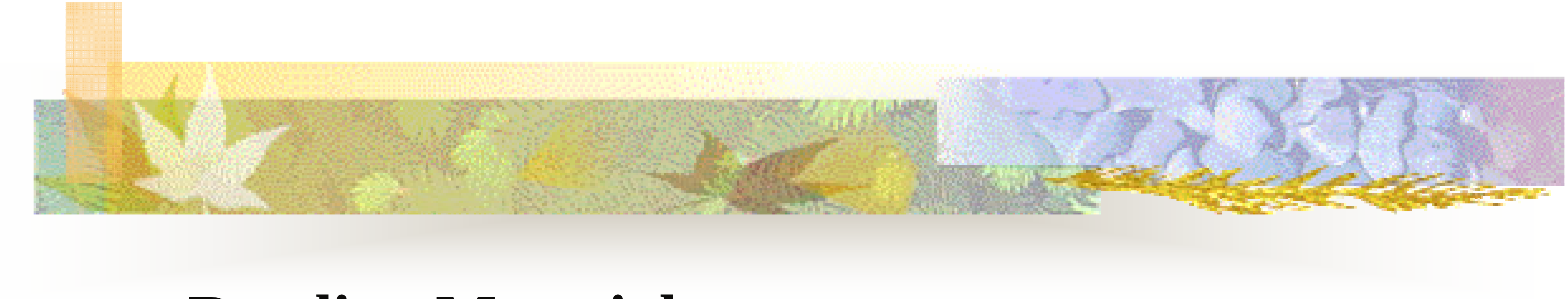
Introduction to Modern Control Theory

- MM 11: Introduction to State-Space Method
- MM 12: Control Design for Full State Feedback
- MM 13: Estimator Design
- MM 14: Introduction of the Reference Input
- MM 15: Integral Control and Robust Tracking

MM1. Introduction to State-Space Method

1. **What's state-space method?**
2. **How to get the state-space description?**
3. **Property Analysis Based on SS Models**



Reading Material: FC: p.469-492, DC: p.101-110

Execercise: see the distributed paper

1. What's the State-Space Method?

In the **state-space method** the differential equations describing a dynamic system are organized as a set of first-order differential equations in the vector-valued state, and the solution is visualized as a trajectory of this state vector in space.

Advantages:

- **To** study more general models
- To deal with multile input and multiple output systems
- To connect internal and external descriptions
- see page 469-472 of FC.....

1.1. State-Space Description

The **state-space representation** is given by the equations:

$$\frac{dX(t)}{dt} = AX(t) + Bu(t) \quad \text{State equation}$$

$$Y(t) = CX(t) + Du(t) \quad \text{Output equation}$$

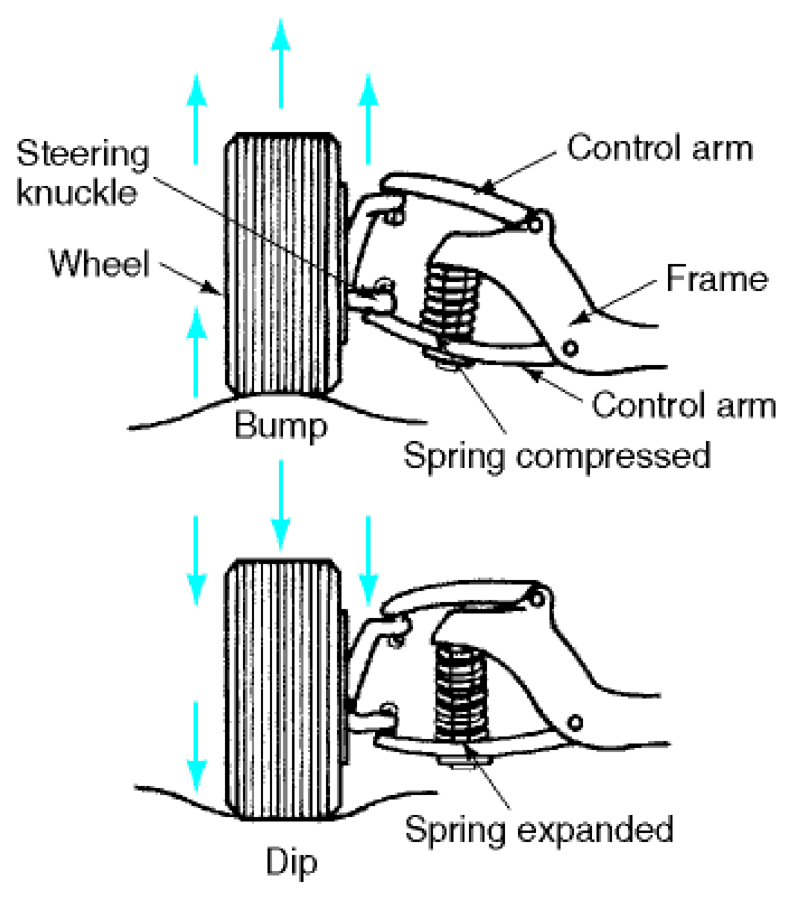
- where X(t) is an nx1 vector representing the state (e.g., position and velocity variables in mechanical systems)
- u(t) is a scalar representing the input (e.g., commonly a force or torque in mechanical systems), and
- y(t) is a scalar representing the output.
- The matrices A (nxn), B (nx1), and C (1xn) determine the relationships between the state and input and output variables.
- State space representation can also be used for systems with multiple inputs and outputs (MIMO), General description

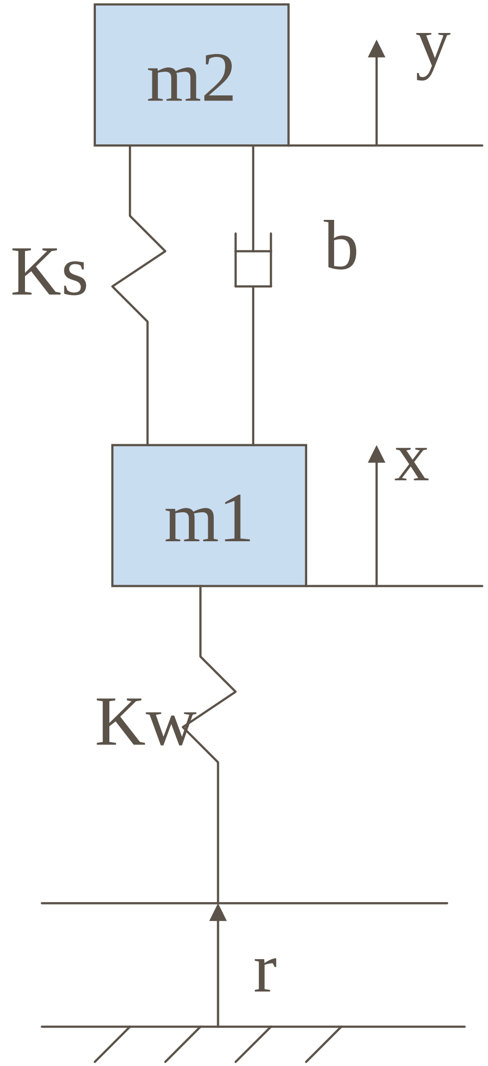
2. How to get a State Space model?

- **2.1 Via modelling techniques**
- 2.2 Via transfer functions
- 2.3 Via other SS descriptions
- 2.4 Via system identification
- ....

2.1 SS Description from Modeling

Example 1: A suspension model





Newton's law for translational motion

2.1.1 SS Description of Suspension system

- Second-order differential equations

$$\ddot{x} + \frac{b}{m_1}(\dot{x} - \dot{y}) + \frac{K_s}{m_1}(x - y) + \frac{K_w}{m_1}x = \frac{K_w}{m_1}r$$

$$\ddot{y} + \frac{b}{m_2}(\dot{y} - \dot{x}) + \frac{K_s}{m_2}(y - x) = 0$$

- State-space description

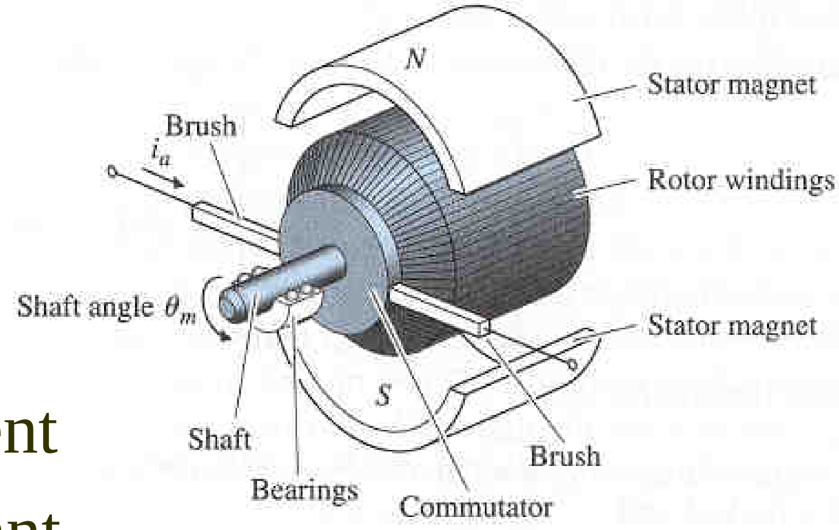
$$\dot{X} = \begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_s}{m_1} - \frac{K_w}{m_1} & -\frac{b}{m_1} & \frac{K_s}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_s}{m_2} & \frac{b}{m_2} & -\frac{K_s}{m_2} & -\frac{b}{m_2} \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_w}{m_1} \\ 0 \\ 0 \end{bmatrix} r$$

2.1.2 Modeling a DC Motor

- Working mechanism of a DC motor

$$T = K_t i_a$$

$$e = K_e \dot{\theta}_m$$



K_t torque constant **i_a** armature current

K_e electromotive force (emf) constant

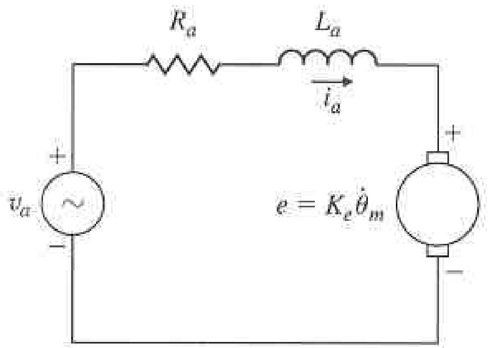
- Differential equation description

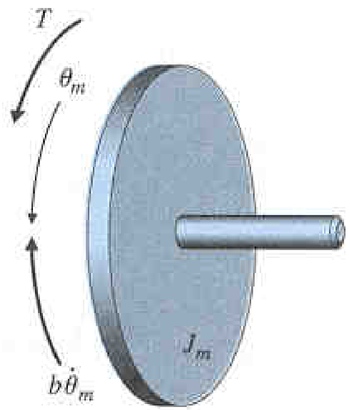
$$J_m \ddot{\theta}_m + b\dot{\theta}_m = K_t i_a$$

$$L_a \frac{di_a}{dt} + R_a i_a = v_a - K_e \dot{\theta}_m$$

simplified :

$$J_m \ddot{\theta}_m + \left(b + \frac{K_t K_e}{R_a}\right)\dot{\theta}_m = \frac{K_t}{R_a} v_a$$





See FC p.47-49

2. How to get a State Space model?

- 2.1 Via modelling techniques
- **2.2 Via transfer functions**
- 2.3 Via other SS descriptions
- 2.4 Via system identification
- ....

2.2.1 Control Canonical Realization

- Transfer function description:

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$$

- **Control canonical SS form:**

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX + DU \end{cases}$$

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_n \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$C = [b_1 \quad b_2 \quad \cdots \quad b_n], \quad D = 0$$

$$\text{where} \quad b(s) = b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n,$$

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n$$

2.2.2 Modal Canonical Realization

Example: $G(s) = \frac{b(s)}{a(s)} = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$

- **Modal canonical form**

$$\begin{cases} \dot{X} = A_m X + B_m U \\ Y = C_m X + D_m U \end{cases}$$

$$A_m = \begin{bmatrix} -4 & 0 \\ 0 & -3 \end{bmatrix}, \quad B_m = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$C_m = \begin{bmatrix} 2 & -1 \end{bmatrix}, \quad D_m = 0$$

System poles appear as the elements along the diagonal of A_m
The numerator terms in the partial-fraction expansion appear in the C_m

2.2.3 Matlab Implementation

- **Control canonical:** [A,B,C,D] = tf2ss(NUM,DEN)
- **Modal canonical form:** CSYS = canon(SYS,TYPE)
  - csys = canon(sys,'modal') returns a realization csys in modal form, that is, where the real eigenvalues appear on the diagonal of the matrix and the complex conjugate eigenvalues appear in 2-by-2 blocks on the diagonal of A.
  - TYPE: 'companion':

$$G(s) = \frac{b(s)}{a(s)} = \frac{s+2}{s^2+7s+12} = \frac{2}{s+4} + \frac{-1}{s+3}$$

```
Num=[1 2]; Den=[1 7 12];
[A,B,C,D]=tf2ss(Num,Den)

Sys=ss(A,B,C,D)
```

```
Csys1=canon(Sys, 'modal')
Csys2=canon(Sys, 'companion')
```

2. How to get a State Space model?

- 2.1 Via modelling techniques
- 2.2 Via transfer functions
- **2.3 Via other SS descriptions**
- 2.4 Via system identification
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2.3 Transformations between SS Models

- **Principle:**

Given a state-space model `sys` with equations

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

performs the similarity transformation $\bar{x} = Tx$ on the state vector x and produces the equivalent state-space model `sysT` with equations.

$$\dot{\bar{x}} = TAT^{-1}\bar{x} + TBu$$
$$y = CT^{-1}\bar{x} + Du$$

State coordinate transformation for state-space models

sysT = ss2ss(sys,T)

2.3.1 Transform to Control Canonical Form

- **Step one:** computer the controllability matrix

  $T_C=[A \ \ AB \ \ A^2B \ \ .... \ A^{n-1}B\ ]$ (ctrb(A,B), ctrb(sys))

- **Step two:** computer the row t_n through

  $t_n=[\,0 \quad 0 \,....\, 1]T_C^{-1}$

- **Step three:** computer the entire transform matrix

$$T^{-1}=\begin{bmatrix} t_n A^{n-1} \\ t_n A^{n-2} \\ \vdots \\ t_n \end{bmatrix}$$

- **Step four:** computer new system matrices using **T** and $\mathbf{T^{-1}}$

$$A_c = T^{-1}AT, \quad B_c = T^{-1}B, \quad C_c = CT, \quad D_c = D$$

2.3.2 Transform to Modal Canonical Form

- Eigenvalues and eigenvectors of matrix A

$$\lambda_i \nu_i = A \nu_i \qquad \text{for } i=1, 2, \ldots n$$

- Transform to modal canonical form

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX + DU \end{cases} \Rightarrow \begin{cases} \dot{X} = A_m X + B_m U \\ Y = C_m X + D_m U \end{cases}$$

  - **Step 1:** calculate the eigenvalues and eigenvectors of A
  - **Step 2:** construct the transform matrix with paramters α_i

$$T = [\alpha_1 \nu_1 \; \alpha_2 \nu_2 \; \ldots\ldots \; \alpha_n \nu_n]$$

  - **Step 3:** select the scale factors α_i of the eigenvectors such that all the elements of B_m are unity: $B_m = T^{-1}B$
  - **Step 4:** use the determined **T** to calculate

$$A_m = T^{-1}AT, \quad B_m = T^{-1}B, \quad C_m = CT, \quad D_m = D$$

2. How to get a State Space model?

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- ....

2.4 System Identification for SS models

- **What is System Identification?**

  System Identification allows you to build mathematical models of a dynamic system based on measured data.

- **How is that done?**

  Essentially by adjusting parameters within a given model until its output coincides as well as possible with the measured output.

- **What's the CAD tool?**
- **ident in Matlab…**
- **System identification course in IRS7 semester**

MM11. Introduction to State-Space Method

1. What's state-space method?
2. How to get the state-space description?
3. Property Analysis Based on SS Models

3.1 Analyses Based on SS Models

- Eigenvalues of the system eig(A)
- Transmission zeros of the system tzero(A,B,C,D)

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX + DU \end{cases} \Rightarrow \det \begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} = 0$$

- Transfer function description of the system
- Stability criterion
- MIMO systems...
- Simulation a SS system

3.2 Transfer Function from SS Models

- Transform function

[A,B,C,D] = tf2ss(NUM,DEN)

[NUM,DEN] = ss2tf(A,B,C,D)

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX + DU \end{cases} \Rightarrow G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

State-space description | fequency response description

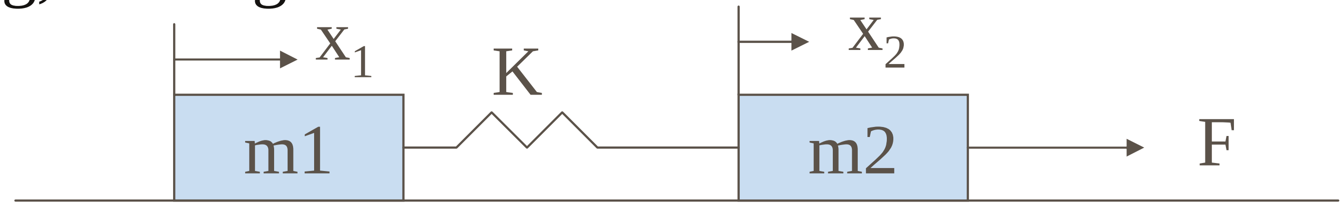
$$\begin{cases} \dot{X} = AX + BU \\ Y = CX + DU \end{cases} \Rightarrow G(s) = \frac{\det\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix}}{\det(sI - A)}$$

Eigenvalues of A
eig(A)
⇄
Poles of G(s)
roots(denominator)

Transmission zero
tzero(A,B,C,D)
⇄
Zeros of G(s)
roots(numerator)

Exercise One (I)

- **Exercise 1:** consider two masses connected via a lossless spring, moving at a surface with no fricition



Assume m1=m2=1, K=1,

- (1.1) derive the state space model of this system, assuming that the force F is input, and position x_1 is output;
- (1.2) computer the eigenvalues of this system and draw out their location in the complex plane;
- (1.3) computer the transfer function of this system based on the state space equations;
- (1.4) find out the corresponding discrete time system description using a zero order hold and sampling period Ts=1.

Exercise One (II)

- **Exercise 2:** consider the following system

$$\dot{X} = \begin{bmatrix} 7 & -9 \\ 6 & -8 \end{bmatrix} X + \begin{bmatrix} 4 \\ 3 \end{bmatrix} u$$
$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} X$$

  - (2.1) Rewrite above system in modal canonical form;
  - (2.2) Rewrite above system in control canonical form;
  - (2.3) Use matlab functions canon(sys) and [num,den]=ss2tf(A,B,C,D), [Ac,Bc,Cc,Dc]=tf2ss(num,den) to check your results.