# M2.The Z-Transform and its Properties

**Reading Material:** Page 94-126 of chapter 3

# What did we talk about in MM1?

**MM1 - Discrete-Time Signal and System**

# MM1: Discrete-Time Signals: Sequences

- ***Continuous-time signals*** are defined along a continuum of times (analog signals)
- ***Discrete-time signals* are defined at discrete times**
- ***Digital signals*** are those for which both time and amplitude are discrete
- Delayed sequence, $y[n]=x[n-n_0]$, $n_0$ is an integer
- Discrete-time impulse,

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

- Unit step sequence,

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

- *Sinusoid sequences*, $\boldsymbol{x[n]=A\cos(\omega_0 n+\phi), -\infty < n < \infty}$

# MM1: Discrete-Time Systems

- A **linear** system:
  $T\{ax1[n]+bx2[n]\}=T\{ax1[n]\}+T\{bx2[n]\}=ay1[n]+by2[n]$
- A **time-invariant** system:
  $T\{\}: x[n] \rightarrow y[n], \quad \Rightarrow \quad T\{\}: x[n-n_d] \rightarrow y[n-n_d]$, for any $n_d$
- A **causal** system
- **BIBO stability**


- **LTI** system: $\mathbf{y[n]=\Sigma_{k=-\infty}^{\infty} x[k]\, h[n-k]}$
- Convolution sum: $\mathbf{y[n]=x[n]*h[n]}$
- FIR: Finite-duration impulse response systems
- IIR: Infinite-duration impulse response systems

# How about Exercise One?

- Problem 2.24  on page75 of the textbook;
- Problem 2.36  on page78 of the textbook.

# MM2 - The Z-Transform

**Reading Material:** Page 94-126 of chapter 3

# Review of Laplace Transform

- Laplace transform for the continuous-time signals and systems:

$$X(s) = L(x(t)) = \int_0^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\delta - j\infty}^{\delta + j\infty} X(s)e^{st}ds, \quad t \geq 0, \delta \geq \delta_0$$

- Benefits:
  - Turn the differential equation into algebraic equation
  - Deal with discontinuous signals
  - System's transient and steady-state analysis

# Review of Laplace Transform (continue)

- **Existing Theorem: x(t)** is Laplace-transformable if
  - **x(t)** is piecewise continuous over every finite interval
  - **x(t)** is exponential order, i.e., $|x(t)|e^{-at}<\infty$
- **Properties:**
  - Linearity $L(ax_1(t)+bx_2(t))=aX_1(s)+bX_2(s)$
  - Shift in time $L(x(t-a))=e^{-as} X(s)$
  - Complex differentiation $L(tx(t))=-d(X(s))/ds$
  - Real differentiation $L(d(x(t))/dt)=sX(s)-X(0)$
  - Final value, initial value ….
- **Applications**
  - Solve the differential equations …...

# What's the Z-transform?

- The z-transform for the discrete-time signals is the counterpart of the Laplace transform for continuous-time signals
- The z-transform is a generalization of the Fourier transform
- In the analysis problems, especially for the discrete-time domain, the z-transform is more convenient than the Fourier transform

# Definition of the Z-Transform

- The z-transform of a sequence **x[n]** is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- Relation with Fourier transform, **$z=re^{j\omega}$**

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-jn\omega}$$

The Fourier transform corresponds the
z-transform on the unit circle in the z-plane

**Convergence of the series?**

# Region of Convergence (ROC)

- For any given sequence, **ROC** is a set of values of z for which the z-transform converges
- ROC depends only on **|z|**, i.e. **|X(z)|<∞,** if

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

- The z-transform and all its derivatives are continuous functions of z within the ROC **( ⇐ Laurent series)**

- A **rational** function **X(z)** is a ratio of two polynomials in z:

**X(z)=P(z)/Q(z)**

Zeros of X(z); poles of X(z)

# Computation of the z-transform

- Right-sided exponential sequence (example 3.1 pp.98)

  $$a^n u[n] \Leftrightarrow 1/(1-az^{-1}), \quad |z|>|a|$$

- Left-sided exponential sequence (example 3.2, pp.99)

  $$-a^n u[-n-1] \Leftrightarrow 1/(1-az^{-1}), \quad |z|<|a|$$

- Sum of two exponential sequences

  $$x[n]=(1/2)^n u[n]+(-1/3)^n u[n]$$

  - According to definition (example 3.3, pp.101)
  - Using the linearity of z-transform (example 3.4, pp.102)
  - The ROC is the intersection of the individual ROCs

# Computation of the z-transform(continue)

- Two-sided exponential sequence (example 3.5, pp.102)

  $x[n]=(-1/3)^n u[n]-(1/2)^n u[-n-1]$

- Finite-length sequence (example 3.6, pp.103)

$$x[n]=\begin{cases} a^n & 0\le n\le N-1 \\ 0 & otherwise \end{cases} \quad \Leftrightarrow \quad X(z)=\frac{1}{z^{N-1}}\frac{z^N-a^N}{z-a}$$

  - The ROC is the entire z-plane except the origin (**z=0**)
  - The **N** zeros are $z_k=ae^{j(2k\pi/N)}$, **k=0,1,…N-1**

**The z-transform includes a algebraic expression and a ROC**

**TABLE 3.1** SOME COMMON $z$-TRANSFORM PAIRS

| Sequence | Transform | ROC |
|---|---|---|
| 1. $\delta[n]$ | $1$ | All $z$ |
| 2. $u[n]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert > 1$ |
| 3. $-u[-n-1]$ | $\dfrac{1}{1-z^{-1}}$ | $\lvert z\rvert < 1$ |
| 4. $\delta[n-m]$ | $z^{-m}$ | All $z$ except 0 (if $m>0$) or $\infty$ (if $m<0$) |
| 5. $a^n u[n]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 6. $-a^n u[-n-1]$ | $\dfrac{1}{1-az^{-1}}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 7. $na^n u[n]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| 8. $-na^n u[-n-1]$ | $\dfrac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert < \lvert a\rvert$ |
| 9. $[\cos\omega_0 n]u[n]$ | $\dfrac{1-[\cos\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 10. $[\sin\omega_0 n]u[n]$ | $\dfrac{[\sin\omega_0]z^{-1}}{1-[2\cos\omega_0]z^{-1}+z^{-2}}$ | $\lvert z\rvert > 1$ |
| 11. $[r^n\cos\omega_0 n]u[n]$ | $\dfrac{1-[r\cos\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 12. $[r^n\sin\omega_0 n]u[n]$ | $\dfrac{[r\sin\omega_0]z^{-1}}{1-[2r\cos\omega_0]z^{-1}+r^2z^{-2}}$ | $\lvert z\rvert > r$ |
| 13. $\begin{cases} a^n, & 0\le n\le N-1,\\ 0, & \text{otherwise}\end{cases}$ | $\dfrac{1-a^N z^{-N}}{1-az^{-1}}$ | $\lvert z\rvert > 0$ |

**(page104)**

# Properties of ROC

- **P1:** The ROC is a ring or disk in the z-plane centered at the origin, i.e., $0 \le r_R \le |z| \le r_L \le \infty$
- **P2:** The Fourier transform of **x[n]** converges absolutely iff the ROC of the z-transform of **x[n]** includes the unit circle in the z-plane
- **P3:** The ROC can not contain any poles
- **P4:** If **x[n]** is a finite-duration sequence, then the ROC is the entire z-plane, except possibly **z=0** or **z= ∞**

$$X(z) = \sum_{n=N_1}^{N_2} x[n] z^{-n}$$

  - **If $N_1<0$, the exception will be z= ∞; if $N_2>0$, the exception will be z= 0;if $N_2>0$ and $N_1<0$, the exception will be both.**

# Properties of ROC (continue)

- **P5:** If **x[n]** is a right-sided sequence, then the ROC extends outward from the outermost finite pole to (and possibly including) $z=\infty$
  - **If the sequence has nonzero values for negative values of n, the ROC will not include $z=\infty$**
- **P6:** If **x[n]** is a left-sided sequence, then the ROC extends inward from the innermost nonzero pole to (and possibly including) $z=0$
  - **If the sequence has nonzero values for positive values of n, the ROC will not include $z=0$**
- **P7:** If **x[n]** is a two-sided sequence, then the ROC will contain a ring in the z-plane, bounded on the interior and exterior by a pole and does not contain any poles
- The ROC must be a connected region

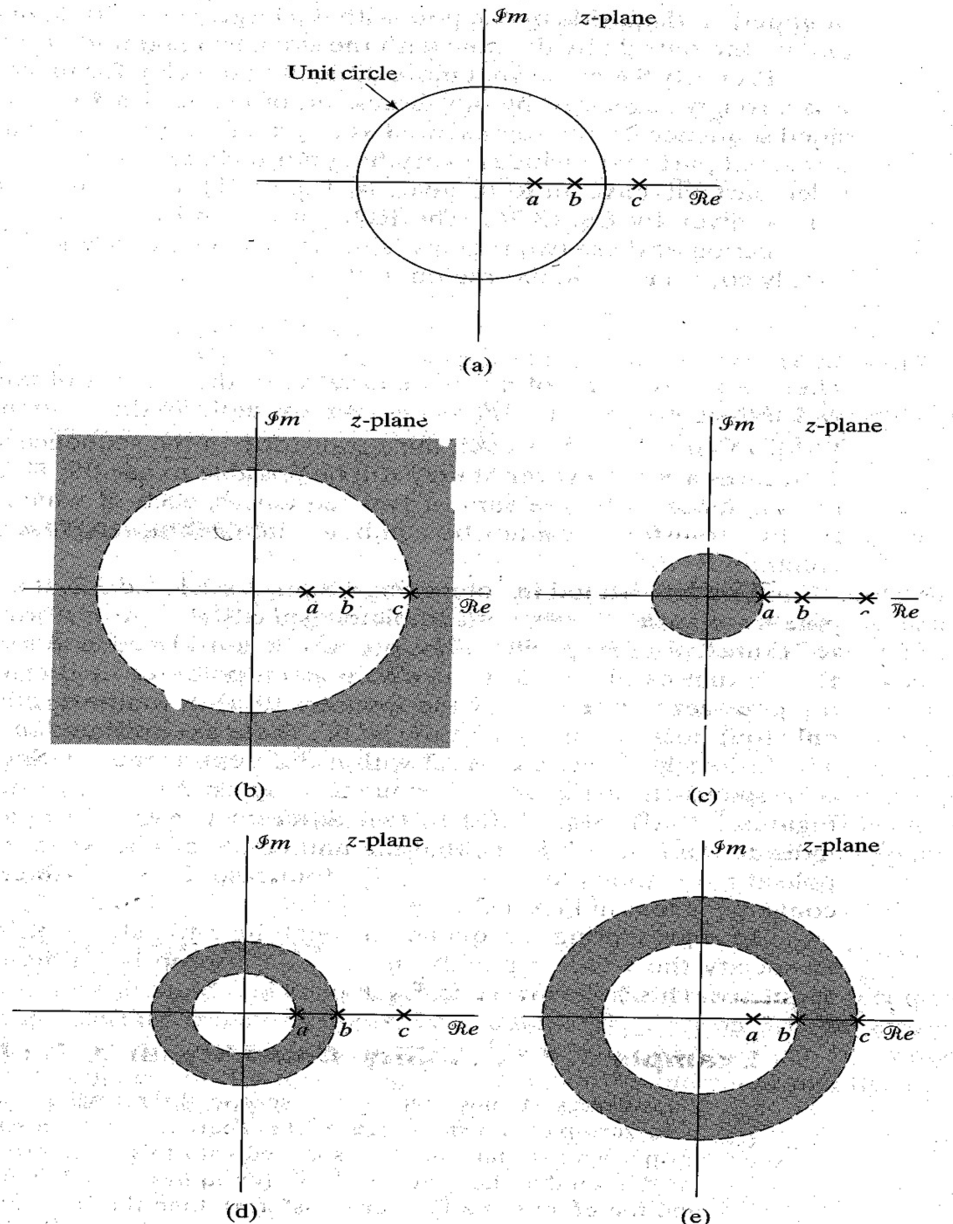


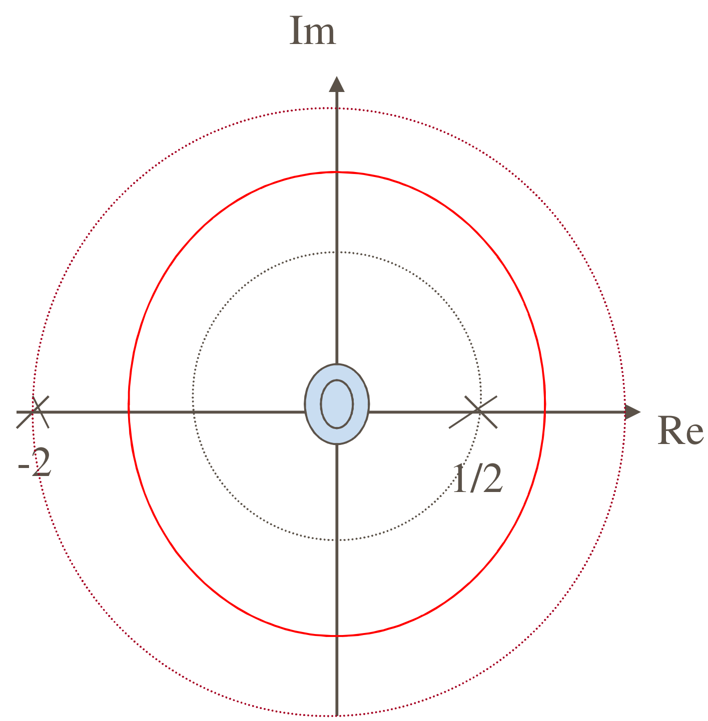

**Figure 3.10** Examples of four $z$- transforms with the same pole–zero locations, illustrating the different possibilities for the region of convergence. Each ROC corresponds to a different sequence: (b) to a right-sided sequence, (c) to a left-sided sequence, (d) to a two-sided sequence, and (e) to a two-sided sequence.

# Determining ROC by System Properties

- The ROC can be specify through some time-domain properties
- Example 3.7 pp.110
- If the system is stable, then the ROC is …. **1/2<|z|<2**

  Is the system causal in this case? ….. **No!! Why…**
- If the system is causal, then the ROC is …. **|z|>2**

  Is the system stable in this case? ….. **No!! Why…**




Pole-zero plot for the system function

# Z-Transform Properties

- Linearity

  $ax_1[n]+bx_2[n] \Leftrightarrow aX_1(z)+bX_2(z)$, **ROC** contains $R_1 \cap R_2$

  - If there is no pole-zero cancellation, **ROC=** $R_1 \cap R_2$
  - If there is the pole-zero cancellation, **ROC** $\geq R_1 \cap R_2$ possibly
  - **Example 3.6, p.103** $x[n] = a^n u[n] - a^n u[n-N] \quad \Leftrightarrow \quad X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$

- Time shifting

  $x[n-n_0] \Leftrightarrow z^{-n_0} X(z)$ , **ROC=** $R_x$ (except for the possible addition or deletion of $z=0$ or $z=\infty$ )

- Exponential multiplication

  $z_0^n x[n] \Leftrightarrow X(z/z_0)$, **ROC=** $|z_0| R_x$

  - **Example 3.15, pp.121**

$$x[n] = r^n \cos(\omega_0 n) u[n] \quad \Leftrightarrow \quad X(z) = \frac{1 - r\cos\omega_0 z^{-1}}{1 - 2r\cos\omega_0 z^{-1} + r^2 z^{-2}}$$

# Z-Transform Properties (continue 1…)

- Differentiation of **X(z)**

  $$nx[n] \Leftrightarrow -z dX(z)/dz, \quad ROC = R_x$$

  - **Example 3.16, pp.122**

  $$\log(1 + az^{-1}), \; |z| > |a| \Leftrightarrow x[n] = (-1)^{n+1} \frac{a^n}{n} u[n-1]$$

- Conjugation of a complex sequence

  $$x^*[n] \Leftrightarrow X^*(z^*), \quad ROC = R_x$$

- Time reversal

  $$x^*[-n] \Leftrightarrow X^*(1/z^*), \quad ROC = 1/R_x$$

See Example 3.18… pp.124

# Z-Transform Properties (continue 2…)

- Convolution of sequences

  $x_1[n]*x_2[n] \Leftrightarrow X_1(z)X_2(z)$, **ROC** contains $R_1 \cap R_2$

  ***Can you prove that …*** ***see page 124***

- Initial-value theorem: If **x[n]** is zero for **n<0** (causal), then

  $$x[0]=\lim_{z\to\infty} X(z)$$

- Summary of z-transform properties – **table 3.2** (pp.126)

**TABLE 3.2** SOME *z*-TRANSFORM PROPERTIES

| Property Number | Section Reference | Sequence | Transform | ROC |
|---|---|---|---|---|
| | | $x[n]$ | $X(z)$ | $R_x$ |
| | | $x_1[n]$ | $X_1(z)$ | $R_{x_1}$ |
| | | $x_2[n]$ | $X_2(z)$ | $R_{x_2}$ |
| 1 | 3.4.1 | $ax_1[n] + bx_2[n]$ | $aX_1(z) + bX_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| 2 | 3.4.2 | $x[n - n_0]$ | $z^{-n_0}X(z)$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| 3 | 3.4.3 | $z_0^n x[n]$ | $X(z/z_0)$ | $\lvert z_0\rvert R_x$ |
| 4 | 3.4.4 | $nx[n]$ | $-z\dfrac{dX(z)}{dz}$ | $R_x$, except for the possible addition or deletion of the origin or $\infty$ |
| 5 | 3.4.5 | $x^*[n]$ | $X^*(z^*)$ | $R_x$ |
| 6 | | $\mathcal{R}e\{x[n]\}$ | $\frac{1}{2}[X(z) + X^*(z^*)]$ | Contains $R_x$ |
| 7 | | $\mathcal{J}m\{x[n]\}$ | $\frac{1}{2j}[X(z) - X^*(z^*)]$ | Contains $R_x$ |
| 8 | 3.4.6 | $x^*[-n]$ | $X^*(1/z^*)$ | $1/R_x$ |
| 9 | 3.4.7 | $x_1[n] * x_2[n]$ | $X_1(z)X_2(z)$ | Contains $R_{x_1} \cap R_{x_2}$ |
| 10 | 3.4.8 | Initial-value theorem: $x[n] = 0, \quad n < 0$ | $\lim_{z\to\infty} X(z) = x[0]$ | |

# Exercise Two

- See the distributed paper