M4. Fourier Transform and Its Properties

Reading material: p.40-65
The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- ROC depends only on $|z|$, i.e. $|X(z)|<\infty$, if

$$\sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n} < \infty$$

- The z-transform includes a algebraic expression and ROC

e.g., Right-sided sequence $a^{n}u[n] \Leftrightarrow 1/(1-az^{-1})$, $|z|>|a|

Left-sided sequence $-a^{n}u[-n-1] \Leftrightarrow 1/(1-az^{-1})$, $|z|<|a|$
Z-Transform

- The z-transform of a sequence $x[n]$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- ROC depends only on $|z|$, i.e. $|X(z)| < \infty$, if

- The z-transform includes a **algebraic expression and ROC**
- Right-sided exponential sequence (example 3.1 pp.98)
  \[ a^n u[n] \Leftrightarrow \frac{1}{1-az^{-1}} , \quad |z|>|a| \]
- Left-sided exponential sequence (example 3.2, pp.99)
  \[ -a^n u[-n-1] \Leftrightarrow \frac{1}{1-az^{-1}} , \quad |z|<|a| \]
- The ROC properties, and
- The z-transform properties
- Inverse z-transform and its properties
Laplace Transform & Fourier Transform

- Laplace transform for the continuous-time signals

\[ X(s) = L(x(t)) = \int_{0}^{\infty} x(t)e^{-st} \, dt \]

\[ x(t) = \frac{1}{2\pi j} \int_{\delta-j\infty}^{\delta+j\infty} X(s)e^{st} \, ds, \quad t \geq 0, \quad \delta \geq \delta_0 \]

- Fourier transform for the continuous-time signals

\[ X(j\Omega) = \int_{0}^{\infty} x(t)e^{-j\Omega t} \, dt \]

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega)e^{j\Omega t} \, d\Omega, \]
Motivation for FT

- Tuning fork problem
- "pure tone signal" and music – a sum of pure tone signals at different frequencies!
- Distinguish two pure tone signals
- Signal spectrum analysis
Complex exponential sequences like $e^{j\omega n}$ are eigenfunctions for the LTI systems, i.e., $y[n] = H(e^{j\omega}) e^{j\omega n}$

The system’s frequency response

$$H(e^{j\omega}) = (\sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k})$$

describes the change in complex amplitude of a complex exponential input signal as a function of the frequency $\omega$

- $H(e^{j\omega})$ is periodic with period $2\pi$
- Frequency domain analysis, lowpass filter, highpass filter…

Suddenly Applied Complex Exponential inputs

$x[n] = e^{j\omega u[n]}$, there is $y[n] = y_{ss}[n] + y_t[n]$, where

$$y_{ss} [ n ] = H ( e^{j\omega} ) e^{j\omega n}, \quad y_t [ n ] = - ( \sum_{k = n + 1}^{\infty} h[k] e^{-j\omega k} ) e^{j\omega n}$$
Motivation for Fourier Transform

- If we can get the expression for any arbitrary input sequence as
  \[ x[n] = \sum_k \alpha_k e^{j\omega_k n} \]
  Then we could get
  \[ y[n] = \sum_k \alpha_k H(e^{j\omega_k}) e^{j\omega_k n} \]
- If the suddenly applied complex exponential sequence such as \( x[n] = \sum_k \alpha_k e^{j\omega_k n} u[n] \) is employed, then we possibly have
  \[ y[n] = \sum_k \alpha_k H(e^{j\omega_k}) e^{j\omega_k n} - \sum_k \alpha_k (\sum_{k=n+1}^{\infty} h[k] e^{-j\omega_k k} e^{j\omega_k n}) \]
- If the system is stable, the second part (transient response) will approach zero as \( n \to \infty \)
Many sequences can be represented by a Fourier integral form as

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{jwn} \, d\omega \]

where

\[ X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-jwk} \]

\[ x[n] \] can be regarded as a superposition of infinitesimally small different complex sinusoids

\[ x[n] = \frac{1}{2\pi} \lim_{\Delta\omega \to 0} \sum_{k} \left( \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega_k m} \right) e^{j\omega_k n} \Delta\omega \]
Expressions of Fourier Transform

- \( X(e^{j\omega}) \) can be expressed as the real and imaginary parts
  \[
  X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})
  \]
  Or in polar form as
  \[
  X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})}
  \]

- \( |X(e^{j\omega})| \) is the magnitude (magnitude spectrum, amplitude spectrum)
- \( \angle X(e^{j\omega}) \) is the phase (phase spectrum)
- The principal value of \( \angle X(e^{j\omega}) \), denoted as \( \text{ARG}[X(e^{j\omega})] \) is restricted to the range of \(-\pi < \omega \leq \pi\)
- \( \text{arg}[X(e^{j\omega})] \) refers to a phase function that is a continuous function of \( \omega \) for \( 0 < \omega < \pi \)
Discussion of Fourier Transform

- The **frequency response** of the LTI system is the Fourier transform of the **impulse response**, and vice versa.

\[
H ( e^{j\omega} ) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}
\]

\[
h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H ( e^{j\omega} ) e^{j\omega n} d\omega
\]

- **Problem 1**: Prove that the expression of \( x[n] \) and \( X(e^{j\omega}) \) are inverse of each other (exercise!)

- **Problem 2**: What kind of signals can be expressed as

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X ( e^{j\omega} ) e^{jwn} d\omega
\]
Using Matlab ...

start matlab
Existence of Fourier Transform

- **Absolute summability** is a sufficient Condition for the existence of the Fourier transform (uniform convergence), why?

\[ | X(e^{j\omega}) | = \sum_{k=-\infty}^{\infty} |h[k]| e^{-jwk} \leq \sum_{k=-\infty}^{\infty} |h[k]| e^{-jwk} \leq \sum_{k=-\infty}^{\infty} |h[k]| \leq \infty \]

- Any stable system will have a finite and continuous frequency response, and any FIR system …
Mean-Square Convergent Sequences

- If some sequences are not absolutely summable, but are square summable, i.e.,

\[ \sum_{k=-\infty}^{\infty} |h[k]|^2 < \infty \]

then its Fourier transform expression

\[ X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \]

has the property – mean square convergence

\[ \lim_{M \to \infty} \int_{-\pi}^{\pi} |X(e^{j\omega}) - X_M(e^{j\omega})|^2 \, d\omega = 0 \]

where

\[ X_M(e^{j\omega}) = \sum_{k=-M}^{M} h[k] e^{-j\omega k} \]
Example 2.22 Square-summable ideal lowpass filter

- Frequency response is
  \[ H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \]

- Impulse response is
  \[ h_{lp}[n] = \frac{\sin \omega_c n}{n\pi}, \quad -\infty < n < \infty \] Noncausal

- \( h_{lp}[n] \) is not absolutely summable, but square summable

Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at \( \omega = \omega_c \) is often called the Gibbs phenomenon.
Some Useful Fourier Transforms

- **(Example 2.23 p.53)** The Fourier transform of the constant sequence, i.e., $x[n]=1$ for all $n$, is the periodic impulse train, i.e.,

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega+2\pi r)$$

- **(Example 2.24 p.54)** Fourier transform of the complex exponential sequence i.e., $x[n]=e^{j\omega_0 n}$ for all $n$, is

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0+2\pi r)$$

- The Fourier transform of the step sequence $x[n]=u[n]$ is represented by (p.54)

$$U(e^{j\omega}) = \frac{1}{1-e^{j\omega}} + \sum_{r=-\infty}^{\infty} \pi\delta(\omega+2\pi r)$$

*Can you prove it? …*
## TABLE 2.3 FOURIER TRANSFORM PAIRS

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\delta[n]$</td>
<td>1</td>
</tr>
<tr>
<td>2. $\delta[n-n_0]$</td>
<td>$e^{-j\omega n_0}$</td>
</tr>
<tr>
<td>3. 1 ($-\infty &lt; n &lt; \infty$)</td>
<td>$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega + 2\pi k)$</td>
</tr>
<tr>
<td>4. $a^n u[n]$ ($</td>
<td>a</td>
</tr>
<tr>
<td>5. $u[n]$</td>
<td>$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k)$</td>
</tr>
<tr>
<td>6. $(n + 1)a^n u[n]$ ($</td>
<td>a</td>
</tr>
<tr>
<td>7. $\frac{r^n \sin \omega_p (n+1)}{\sin \omega_p} u[n]$ ($</td>
<td>r</td>
</tr>
<tr>
<td>8. $\frac{\sin \omega_c n}{\pi n}$</td>
<td>$X(e^{j\omega}) = \begin{cases} 1, &amp; \omega &lt; \omega_c, \ 0, &amp; \omega_c &lt;</td>
</tr>
<tr>
<td>9. $x[n] = \begin{cases} 1, &amp; 0 \leq n \leq M \ 0, &amp; \text{otherwise} \end{cases}$</td>
<td>$\frac{\sin[\omega(M + 1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$</td>
</tr>
<tr>
<td>10. $e^{j\omega_0 n}$</td>
<td>$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k)$</td>
</tr>
<tr>
<td>11. $\cos(\omega_0 n + \phi)$</td>
<td>$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi} \delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi} \delta(\omega + \omega_0 + 2\pi k)]$</td>
</tr>
</tbody>
</table>
Symmetry Properties

**Motivation:** Symmetry properties are often very useful for simplifying the solution of the problems

**Some definitions:**
- If $x_e[n] = x_e^*[-n]$, $x_e[n]$ is called **conjugate-symmetric sequence**
- If $x_o[n] = -x_o^*[-n]$, $x_e[n]$ is called **conjugate-antisymmetric sequence**

**Any sequence** can be expressed as a sum of a conjugate-symmetric and conjugate-antisymmetric sequence, i.e.,

$$X[n] = x_e[n] + x_o[n], \quad \text{where}$$

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n])$$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n])$$

- A real sequence that is conjugate-symmetric is called **even sequence**, while which is conjugate-antisymmetric is called **odd sequence** (! an error in p.55 $x_o[n] = -x_o[-n]$)
Symmetry Properties of FT

- A Fourier transform $X(e^{j\omega})$ can be decomposed into a sum of conjugate-symmetric and conjugate-antisymmetric functions, i.e., $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$, where
  
  $$X_e(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) + X^*(e^{-j\omega}))$$  
  $$X_o(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) - X^*(e^{-j\omega}))$$

- If $x[n]$ is a real sequence, the real part $X_R(e^{j\omega})$ of the Fourier transform $X(e^{j\omega})$ is an even function, and the imaginary part $X_I(e^{j\omega})$ is an odd function.

- If $x[n]$ is a real sequence, the magnitude $|X(e^{j\omega})|$ of the Fourier transform $X(e^{j\omega})$ in polar form is an even function, and the phase $\angle X(e^{j\omega})$ is an odd function.

- See example 2.25 page 57….
### TABLE 2.1  SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

<table>
<thead>
<tr>
<th>Sequence $x[n]$</th>
<th>Fourier Transform $X(e^{j\omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $x^*[n]$</td>
<td>$X^*(e^{-j\omega})$</td>
</tr>
<tr>
<td>2. $x^*[-n]$</td>
<td>$X^*(e^{j\omega})$</td>
</tr>
<tr>
<td>3. $\Re{x[n]}$</td>
<td>$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)</td>
</tr>
<tr>
<td>4. $j\Im{x[n]}$</td>
<td>$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)</td>
</tr>
<tr>
<td>5. $x_e[n]$ (conjugate-symmetric part of $x[n]$)</td>
<td>$X_R(e^{j\omega}) = \Re{X(e^{j\omega})}$</td>
</tr>
<tr>
<td>6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$)</td>
<td>$j X_I(e^{j\omega}) = j\Im{X(e^{j\omega})}$</td>
</tr>
</tbody>
</table>

**The following properties apply only when $x[n]$ is real:**

<table>
<thead>
<tr>
<th>Property</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>7. Any real $x[n]$</td>
<td>$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)</td>
</tr>
<tr>
<td>8. Any real $x[n]$</td>
<td>$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)</td>
</tr>
<tr>
<td>9. Any real $x[n]$</td>
<td>$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)</td>
</tr>
<tr>
<td>10. Any real $x[n]$</td>
<td>$</td>
</tr>
<tr>
<td>11. Any real $x[n]$</td>
<td>$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)</td>
</tr>
<tr>
<td>12. $x_e[n]$ (even part of $x[n]$)</td>
<td>$X_R(e^{j\omega})$</td>
</tr>
<tr>
<td>13. $x_o[n]$ (odd part of $x[n]$)</td>
<td>$j X_I(e^{j\omega})$</td>
</tr>
</tbody>
</table>
Fourier Transform Theorems

- **Linearity**
  \[ ax_1[n] + bx_2[n] \iff aX_1(e^{i\omega}) + bX_2(e^{i\omega}) \]

- **Time shifting**
  \[ x[n-n_d] \iff e^{-j\omega n_d} X(e^{i\omega}) \]

- **Frequency shifting**
  \[ e^{j\omega_0 n} x[n] \iff X(e^{j(\omega-\omega_0)}) \]

- **Time reversal**
  \[ x[-n] \iff X^*(e^{i\omega}) \]

- **Differentiation in frequency**
  \[ nx[n] \iff j \left( \frac{dX(e^{i\omega})}{d\omega} \right) \]
Fourier Transform Theorems (continue)

- Parseval’s Theorem

\[ E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega \]

\(|X(e^{j\omega})|^2\) is the energy density spectrum, i.e., it determines how the energy is distributed in the frequency domain.

- The convolution Theorem: if \( y[n] = x[n] * h[n] \), then

\[ Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \]

- The modulation (windowing) theorem: if \( y[n] = x[n]w[n] \), then

\[ Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta \]
<table>
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<th>Sequence</th>
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<tbody>
<tr>
<td>$x[n]$</td>
<td>$X(e^{j\omega})$</td>
</tr>
<tr>
<td>$y[n]$</td>
<td>$Y(e^{j\omega})$</td>
</tr>
</tbody>
</table>

1. $ax[n] + by[n]$  \quad $aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$  \quad ($n_d$ an integer) \quad $e^{-j\omega n_d}X(e^{j\omega})$
3. $e^{j\omega_0 n}x[n]$ \quad $X(e^{j(\omega - \omega_0)})$
4. $x[-n]$ \quad $X(e^{-j\omega})$
5. $nx[n]$ \quad $X^*(e^{j\omega})$ if $x[n]$ real.
6. $x[n] \ast y[n]$ \quad $\frac{j}{2\pi}X(e^{j\omega})$
7. $x[n]y[n]$ \quad $X(e^{j\omega})Y(e^{j\omega})$

Parseval's theorem:
8. $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega}) d\omega$
Exercise Four

Problem 2.32 on page 76 of the textbook.

(optional) Prove that The Fourier transform – equation (2.134) in page 48 of the textbook is the inverse of the (synthesis) equation (2.133) in page 48 of the textbook;