M5. LTI Systems Described by Linear Constant Coefficient Difference Equations

Reading Material: p.34-40, 245-253
Up til now…
we introduced the Fourier and z-transforms and their properties with only brief preview of their use in the analysis of LTI systems

In the following…
We will develop in more detail the representation and analysis of LTI systems using the Fourier and z-transforms
Transforms and Their Properties

- **Z-transform**
  \[ X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \]

- **ROC’s properties** ...

- **System function**
  \[ H(z) = (\sum_{k=-\infty}^{\infty} h[k] z^{-1}) \]

- **Output response**
  \[ Y(z) = H(z)X(z) \]

- **Inverse z-transform**: Inspection, Partial fraction expansion, Power series expansion ...

- **Properties**: linearity, time shifting, time reversal, differentiation, convolution ...

- **Fourier transform**
  \[ X[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{-j\omega n} d\omega \]

- **Eigenfunctions**: \( e^{j\omega n} \)

- **Frequency response**
  \[ H(e^{j\omega}) = (\sum_{k=-\infty}^{\infty} h[k] e^{-jn\omega}) \]

- **Output response**
  \[ Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) \]

- **Properties**: symmetry, linearity, time shifting, time reversal, differentiation, Parseval’s theorem, convolution, modulation ...

\[ X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n})e^{-jn\omega} \]

The Fourier transform corresponds the z-transform on the unit circle in the z-plane.
Related to Rational Functions…

- First time of mention is in the mm2 (ROC discussion) …
  - A rational function $X(z)$ is a ratio of two polynomials in $z$:
    \[ X(z) = \frac{P(z)}{Q(z)}, \]
    zero: $P(c_k) = 0$; poles: $Q(d_k) = 0$

- Second time of mention is in mm3 (inverse z-trans) …
  - Any rational function $X(z)$ can always be expressed as a sum of simpler terms, each of which is tabulated
    \[ X(z) = \frac{b_0}{a_0} \prod_{k=1}^{M} \left(1 - c_k z^{-1}\right) \prod_{k=1}^{N} \left(1 - d_k z^{-1}\right) \iff X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} A_k \left(1 - d_k z^{-1}\right)^{-1} \]
    - Inverse z-transform … \[ x[n] = \sum_{r=0}^{M-N} B_r \delta[n - r] + \sum_{k=1}^{N} A_k d_k^n u[n] \]
    - Example 3.9, page115… \[ X(z) = \frac{1 + 2 z^{-1} + z^{-2}}{1 - \frac{3}{2} z^{-1} + \frac{1}{2} z^{-2}}, \quad |z| > 1 \]

What kind of systems has the z-transform as a rational function?
LTI Systems Described by LCCDE

- Linear constant coefficient difference equations (LCCDE) is used to describe a subclass of LTI systems, which input and output satisfy an $N$th-order difference equation as
\[
\sum_{k=0}^{N} a_k y[n-k] = \sum_{m=0}^{M} b_m x[n-m]
\]

- It gives a better understanding of how to implement the LTI systems, such as

\[x[n] \xrightarrow{Z^{-1}} b_0 \xrightarrow{Z^{-1}} b_1 \xrightarrow{Z^{-1}} \ldots \xrightarrow{Z^{-1}} b_M \quad \text{and} \quad \xrightarrow{Z^{-1}} -a_1 \xrightarrow{Z^{-1}} \ldots \xrightarrow{Z^{-1}} -a_N \]

\[y[n] \xrightarrow{Z^{-1}} \]

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I. Discrete-Time Signals and Systems
Examples of LCCDE Systems

- **Example 2.14** Recursive Representation of accumulator

  \[ y[n] = \sum_{k=-\infty}^{n} x[k] \iff y[n] - y[n-1] = x[n] \iff y[n] = y[n-1] + x[n] \]

- **Example 2.15** the moving-average system:

  - IR is \( h[n] = \frac{1}{(M_2 + 1)}(u[n] - u[n-M_2-1]) \), then
    
    (a) \[ y[n] = \frac{1}{(M_2 + 1)} \sum_{k=0}^{M_2} x[n-k] \]

  Also there is

  - \( h[n] = \frac{1}{(M_2 + 1)}(\delta[n] - \delta[n-M_2-1]) \cdot u[n] \), then
    
    (b) \[ y[n] - y[n-1] = \frac{1}{(M_2 + 1)} (x[n] - x[n-M_2-1]) \]

  The difference equation representations of the LTI systems is not unique!!!
Homogeneous Solutions

For a given input $x_p[n]$, assume $y_p[n]$ is the corresponding output, so that there is

$$\sum_{k=0}^{N} a_k y_p[n-k] = \sum_{m=0}^{M} b_m x_p[n-m]$$

then the same equation with the same input is satisfied by any output of the form

$$y[n] = y_p[n] + y_h[n]$$

where $y_h[n]$ is any solution to the homogeneous equation

$$\sum_{k=0}^{N} a_k y_h[n-k] = 0$$

$y_h[n]$ is called the homogeneous solution
The homogeneous solution $y_h[n]$ has the form

$$y_h[n] = \sum_{m=1}^{N} A_m z_m^n$$

Through $\sum_{k=0}^{N} a_k z_m^{-k}=0$, $N$ coefficients $z_m$ can be determined, while $N$ coefficients $A_m$ still need to be determined, i.e., a set of $N$ auxiliary conditions is required for the unique specification of $y[n]$ for a given $x[n]$.

If a system is characterized by LCCDE and is further specified to be linear, time-invariant, and causal, then the solution is unique. In this case, the auxiliary conditions are referred to as initial-rest conditions (IRC).

IRC means if the input $x[n]$ is zero for $n$ less than some $n_0$, then $y[n]$ is always zero for $n$ less than $n_0$. 
Summary for LCCDE Description

- The output for a given input is not uniquely specified, auxiliary conditions are required.
- If the auxiliary information is in the form of $N$ sequential values of the output, **later values** can be obtained by rearranging the LCCDE as a recursive relation running forward in $n$, such as

\[
y[n] = - \sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{k=0}^{M} b_k x[n-k]
\]

- **Prior values** can be obtained by rearranging the LCCDE as a recursive relation running backward in $n$, like

\[
y[n-N] = - \sum_{k=0}^{N-1} \frac{a_k}{a_N} y[n-k] + \sum_{k=0}^{M} b_k x[n-k]
\]

- If the system is **initially at rest**, then the system will be linear, time-invariant, and causal.
Example 2.16 Recursive Computation

(...page 37-38...)

- LCCDE description
  \[ y[n] = ay[n-1] + x[n] \]
- Input \( x[n] = K\delta[n] \)
- Auxiliary condition \( y[-1] = c \)
- Recursive computation for \( n > -1 \) ... and \( n < -1 \)...
- If the auxiliary is the initial-rest condition with \( y[-1] = c \), how about the result? (see page 39)
- Can you get the z-transform of the LCCDE?
System Functions of LCCDE Systems

\[ \sum_{k=0}^{N} a_k y[n - k] = \sum_{m=0}^{M} b_m x[n - m] \]

\[ \Downarrow \quad \text{linearity and time-invariance} \]

\[ \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{m=0}^{M} b_m z^{-m} X(z) \]

\[ \Downarrow \quad \text{ROCs of } Y(z) \text{ and } X(z) \text{ overlap} \]

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0}{a_0} \prod_{k=1}^{M} \left(1 - c_k z^{-1}\right) \prod_{k=1}^{N} \left(1 - d_k z^{-1}\right) \]
Stability, Causality of LCCDE Systems

- The system is **causal**
  - $h[n]$ is **right-sided** sequence
  - The ROC of $H(z)$ is **outside** the outermost pole

- The system is **stable**
  - $h[n]$ is **absolutely summable**
  - The ROC of $H(z)$ includes the **unit circle**

\[ \sum_{n=-\infty}^{\infty} |h[n]|z^{-n} < \infty \quad \text{for} \quad |z| = 1 \]

The LTI system described by LCCDE is both causal and stable, iff the ROC of the corresponding system function is outside the outermost pole and includes the unit circle

Example 5.3 determine the ROC of $y[n]-5/2y[n-1]+y[n-2]=x[n]$
Inverse System of LCCDE System

The inverse system is defined to be the system with system function \( H_i(z) \) such that if it is cascaded with \( H(z) \), the overall effective system function is unity, i.e.,

\[
G(z) = H(z)H_i(z) = 1
\]

The time-domain equivalence is

\[
g[n] = h[n]*h_i[n] = \delta[n]
\]

This implies that

\[
H_i(z) = \frac{1}{H(z)}
\]

Therefore, the inverse of LCCDE system is

\[
H_i(z) = \frac{a_0}{b_0} \frac{\prod_{k=1}^{M} (1 - d_k z^{-1})}{\prod_{k=1}^{N} (1 - c_k z^{-1})}
\]
Inverse System of LCCDE (continue…)

- In order to make inverse system sensible, the ROC of $H_i(z)$ and $H(z)$ must overlap.

- **Example 5.5** $H(z) = (z^{-1} - 0.5)/(1 - 0.9z^{-1})$ with ROC $|z| > 0.9$

- If $H(z)$ is causal, its inverse system will be causal iff the ROC of $H_i(z)$ is $|z| > \max|c_k|$

- If $H(z)$ is stable, its inverse system will be stable iff $\max|c_k| < 1$

- A stable, causal LTI system has a stable and causal inverse iff both zeros and poles of $H(z)$ are inside the unit circle. --------- **minimum-phase system**
FIR, IIR for Rational System Functions

- LCCDE $\Leftrightarrow$ rational system functions
  $\uparrow$ z-transform and its inverse
  Impulse responses

- For a rational function
  $$H(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

- If the system is causal, there is
  $$h[n] = \sum_{r=0}^{M-N} B_r \delta[n - r] + \sum_{k=1}^{N} A_k d_k^n u[n]$$

- If there is at least one nonzero pole of $H(z)$ is not canceled by a zero, then the system is IIR system; otherwise, the system is FIR system
Exercise Five

- Problem 3.23 (b) on page 132 of the textbook
- Problem 5.4 on page 313 of the textbook
- Problem 5.28 (a), (b), (c-i) on page 321 of the textbook