MODELING AND SIMULATION

Euler-lagrange method
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Preface

- What’re Models for systems and signals?
  - Basic concepts
  - Types of models
- How to build a model for a given system?
  - Physical modeling
  - Experimental modeling
- How to simulate a system?
  - Matlab/Simulink tools
- Case studies
- Text-book: Modeling of dynamic systems  Lennart Ljung and Torkel Glad
Preface - Systems and models

- Part one – Models, p13-78
- **System** is defined as an object or a collection of objects whose properties we want to study

- A **model** of a system is a tool we use to answer questions about the system without having to do an experiment
  - Mental model
  - Verbal model
  - Physical model
  - **Mathematical model**
Preface - physical modeling

- Conservation laws
  - Mass balance
  - Energy balance
  - Electronics (Kirchhoff’s laws)

- Constitutive relationships

\[ m \ a = \sum F \]

\[ J \ \frac{d\omega}{dt} = \sum \tau \]

Figur 5: Roterende mekanisk system
DC motor with Permanent Magnet

Modeling a DC Motor

Find a set of equations for the DC motor with the armature driven by the electric circuit shown in Fig. 2.26(a). Assume the rotor has inertia \( J_m \) and friction coefficient \( b \).

Solution. In our analysis we need to include the back emf for the electrical circuit. For the mechanical part of the system we need to include the motor torque in analyzing the rotor. The free-body diagram for the rotor, shown in Fig. 2.26(b), defines the positive direction and shows the two applied torques, \( T \) and \( b \dot{\theta}_m \). Application of Eq. (2.7) yields

\[
J_m \ddot{\theta}_m + b \dot{\theta}_m = K_i i_a.
\]  

(2.42)

KVL loop analysis shows the electrical equation to be

\[
L_a \frac{di_a}{dt} + R_a i_a = v_a - K_e \dot{\theta}_m.
\]  

(2.43)

In most cases the electrical circuit response is much faster than the rotor motion; thus an applied voltage results in essentially an instantaneous change in the current flow. Therefore, it is often possible to neglect the existence of the inductance \( L_a \) in Eq. (2.43). In this case we combine Eqs. (2.42) and (2.43) into one equation to get

\[
J_m \ddot{\theta}_m + \left( b + \frac{K_e K_i}{R_a} \right) \dot{\theta}_m = \frac{K_e}{R_a} v_a.
\]  

(2.44)

From Eq. (2.44) it is clear that the effect of the back emf is indistinguishable from the friction, in that they both act identically to damp the motion of the motor.
Motivation (1)

Euler-lagrange method

???

Energy perspective...
Motivation (2)

- The mathematical model can be obtained from variational principles applied to energy functions!

- There exists a well established common terminology for all type of systems, whether electrical, mechanical, magnetic, etc., by defining energy functions in terms of the generalized coordinates.

- There are a number of different energy functions (e.g. the Lagrangian, the total energy) which can be used as a energy function.

- The variational approach is quite formal analytically and insight into physical processes may be lost. Nevertheless, if the method is properly understood, physical insight can be gained due to the generality of the method.
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EL Method – Generalized coordinate(1)

- For instance, for the specification of a rigid body, we need six coordinates, three for the reference point and three for the orientation.
- A certain minimum number \( n \) of coordinates, called the **degrees of freedom**, is required to specify the configuration.
- Usually, these coordinates are denoted by \( q_i \) and are called generalized coordinates. The coordinate vector

\[
x_i = x_i (q_1, q_2, \ldots, q_n, t)
\]
EL Method – Generalized coordinate(2)

- The choice of the generalized coordinates is usually somewhat arbitrary, but in general each individual energy storage element of the system have a set of generalized coordinates.
- For a dynamic system the generalized coordinates do not completely specify the system and an additional set of dynamic variables equal in number to the generalized coordinates must be used.
- These dynamic variables can be the first time derivatives of the generalized coordinates, the velocities, or can be a second set variables (e.g. the generalized momenta).
EL Method – Energy

- The kinetic energy $T$ in terms of Cartesian coordinates is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^T v_i, \quad \frac{dx_i}{dt} = v_i.$$  

- The Potential energy $V$, e.g., $V=mg\,h$, $1/2kx^2$

- Lagrangian function: $L=T-V$
EL Method – Euler Lagrange eq.

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^e, \quad j = 1, \ldots, n. \]

\( Q_j^e \) are generalized forces acting along the jth generalized axis.
EL Method – Summary

The general procedure:

- Select a suitable set of coordinates to represent the configuration of the system.

- Obtain the kinetic energy $T$ as a function of these coordinates and their time derivatives.

- If the system is conservative, find the potential energy $V$ as a function of the coordinates, or, if the system is not conservative, find the generalized forces $Q_{je}$.

- The differential equations of motion are then given by EL equations.
Example (1)

A set of generalized coordinates $x_a$ and $x_b$ and their associated velocities $v_a$ and $v_b$.

**Kinetic Energy**

$$T = \frac{1}{2} m_1 v_a^2 + \frac{1}{2} m_2 v_b^2.$$

**Potential Energy**

$$V = \frac{1}{2} c_1 x_1^2 + \frac{1}{2} c_2 x_2^2 + \frac{1}{2} c_3 x_3^2 - m_1 g x_1 - m_2 g x_2$$

$x_1 = x_a + l_1$, $x_2 = x_b + l_3 - x_a - l_1 = x_b - x_a + l_2$, $x_3 = x_b + l_3$

*Figure 1: Mechanical example.*
Example (2)

Lagrangian

\[ L = T - V = \]
\[ \frac{1}{2} m_1 v_a^2 + \frac{1}{2} m_2 v_b^2 - \]
\[ -\frac{1}{2} c_1 (x_a + l_1)^2 - \frac{1}{2} c_2 (x_b - x_a + l_2)^2 - \frac{1}{2} c_3 (x_b + l_3)^2 + m_1 g (x_a + l_1) + m_2 g (x_b + l_3 - x_a - l_1) \]

The generalized forces are

\[ Q_1^e = -F_1, \quad Q_2^e = F_2. \]

Figure 1: Mechanical example.
Example (3)

\[ L = T - V = \]
\[ \frac{1}{2} m_1 v_a^2 + \frac{1}{2} m_2 v_b^2 - \]
\[ -\frac{1}{2} c_1 (x_a + l_1)^2 - \frac{1}{2} c_2 (x_b - x_a + l_2)^2 - \frac{1}{2} c_3 (x_b + l_3)^2 + m_1 g (x_a + l_1) + m_2 g (x_b + l_3 - x_a - l_1) \]

\[ \frac{d}{dt} \frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial q_j} = Q_j^e, \quad j = 1, \ldots, n. \]

\[ x_a: \quad m_1 \frac{dv_a}{dt} + c_1 (x_a + l_1) - c_2 (x_b - x_a + l_2) - m_1 g + m_2 g = -F_1 \]

\[ x_b: \quad m_2 \frac{dv_b}{dt} + c_2 (x_b - x_a + l_2) + c_3 (x_b + l_3) - m_2 g = F_2. \]
**Example - Beam & Ball System (1)**

**Introduction:**

System modeling and simulation provide useful and safe mechanisms for initial controller design. The ball and beam system shown below in figure 1 has the control objective of placing the ball anywhere along the beam by varying the motor voltage. With this objective in mind, the system will be investigated and controlled through various modeling, simulation, and controller design steps.

![Diagram of Beam & Ball System](image)

**Figure 1: Ball and Beam Apparatus**
2. Modeling of gantry cranes

We use the Lagrangian approach to derive the equations of motion. It follows from Fig. 3 that the load and trolley position vectors are given by

\[ \mathbf{r}_L = \{x + L \sin(\phi), -L \cos(\phi)\} \quad \text{and} \quad \mathbf{r}_T = \{x, 0\}. \tag{1} \]

Then, the kinetic and potential energies of the whole system are given by

\[ T = \frac{1}{2}m \mathbf{r}_L \cdot \dot{\mathbf{r}}_L + \frac{1}{2}M \mathbf{r}_T \cdot \dot{\mathbf{r}}_T, \tag{2} \]

\[ V = -mgL \cos(\phi). \tag{3} \]

Let the generalized forces corresponding to the generalized displacements \( q = \{x, \phi\} \) be \( \mathbf{F} = \{F_x, 0\} \). Constructing the Lagrangian \( \mathcal{L} = T - V \) and using Lagrange’s equations

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_j} \right) - \frac{\partial \mathcal{L}}{\partial q_j} = F_j, \quad j = 1, 2, \tag{4} \]

we obtain the following equations of motion:

\[ (m + M) \ddot{x} + mL \ddot{\phi} \cos(\phi) + mL \dot{\phi} \sin(\phi) + 2mL \dot{\phi} \cos(\phi) - mL \dot{\phi}^2 \sin(\phi) = F_x, \tag{5} \]

\[ L \ddot{\phi} + g \sin(\phi) + 2L \dot{\phi} + \ddot{x} \cos(\phi) = 0. \tag{6} \]
PID-based Cascade Control

Diagram showing a cascade control system with inputs and outputs labeled as:
- \( r_p \)
- \( \theta \)
- \( \theta_{\text{ref}} \)
- \( e_x \)
- \( e_\theta \)
- \( u \)
- \( y_p \)

Inputs:
- Computer & Control Software
- D to A
- A to D

Outputs:
- Motor Signal
- Beam Angle
- Ball Position

Ball & Beam Experiment
Conclusion

- The application of the Lagrangian formulation is not restricted to mechanical systems.

- The Lagrangian depends on the generalized coordinates $q$, the associated velocities $\dot{q}$, and the time $t$. 