# Stochastic Analysis for Engineers

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Fall 2004, Aalborg University Esbjerg
Course Web: http://www.cs.aue.auc.dk/~yang/course/stoc04.htm



## Objective

- To give students an **understanding** of the description of stochastic signals in order to perform filtering and detection
- To enable students to **apply** estimation and detection methods for simple problems in connection with stationary stochastic processes
- To give students an **understanding** of spectral estimation techniques



## Textbook

- K. Sam Shanmugan and BArthur M. reipohl: "***Random Signals - Detection, Estimation and Data Analysis*** ", John Wiley Sons, Inc., 1988.



## Content

- Discrete linear models
  - Autoregressive (AR) processes
  - Moving average (MA) models
  - Autoregressive moving average (ARMA) models
- Signal detection
  - Hypothesis testing
  - Decision theory and decision rules
  - Binary detection
  - M-ary detection
- Linear minimum mean-square error estimation
  - Linear minimum mean-square error estimators
  - Nonlinear minimum mean-square error estimators
  - Joint Gaussian random variables



## Content (cont'd)

- Discrete time Wiener filters
  - Non-causal discrete time Wiener filters
  - Causal discrete time Wiener filters
- Discrete time Kalman filters
  - Kalman filters
  - Extended Kalman filters
- Parameter estimation of stochastic processes
  - Model-free estimation: mean value, autocorrelation, PSD
  - Model-based estimation for AR, MA, ARMA processes
- Case studies



## Ten Lecture Topics

| Topics | Sections in Shanmugan |
|---|---|
| Response of linear systems to random inputs | 4.1, 4.2, 4.3 |
| Discrete linear stochastic models | 5.2 |
| Detection of known signals (Part 1) | 6.2, 6.3, 6.5 |
| Detection of known signals (Part 2) | 6.2, 6.3, 6.5 |
| Mean-square error filtering and estimation, Wiener filters (Part 1) | 7.2, 7.3<br>7.5 |
| Wiener filters (Part 2) | 7.5, 7.7 |
| Kalman filters (Part 1) | 7.6 |
| Kalman filters (Part 2) | 7.6 |
| Model-free and spectral estimation (Part 1) | 9.3, 9.4 |
| Model-free and spectral estimation (Part 2) | 9.3, 9.4 |

## What have we learned in Sem6?

- **MM1:** Definition and description of stochastic processes
- **MM2:** Special classes of stochastic processes and stationarity
- **MM3:** Autocorrelation and power spectral density functions of WSS processes
- **MM4:** Continuity, differentiation, integration, time averaging and ergodicity
- **MM5:** Response of Linear Systems to Random Signals



## Random Processes

- Let **S** be the sample space of a random experiment and let **t** be a variable that can have values in the set $\Gamma \subset R_1$, the real line. A **real-valued random process X(t), $t \in \Gamma$**, is a measurable function on $\Gamma \times S$ that maps $\Gamma \times S$ onto $R_1$
- A real-valued random process can be described by its nth order distribution function like

$$F_{X(t_1),X(t_2),\ldots X(t_n)}(x_1,x_2,\ldots x_n) = P[X(t_1) \le x_1,\ldots,X(t_n) \le x_n]$$

for all **n** and $t_1, \ldots, t_n \in \Gamma$



## Method of Description

- First and second order characteristics
  - The mean of **X(t)**

$$\mu_X(t) = E\{X(t)\}$$

  - The autocorrelation of **X(t)**
  - $$R_{XX}(t_1,t_2) = E\{X^*(t_1)X(t_2)\}$$
  - The autocovariance of **X(t)**

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X^*(t_1)\mu_X(t_2)\}$$

  - The correlation coefficient of **X(t)**

$$r_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_1)C_{XX}(t_2,t_2)}}$$



## Matlab Calculations

- **randn(m,n)** function generates arrays of random numbers whose elements are normally distributed with mean 0, and variance 1
- **M = mean(A)** returns the mean values of the elements along different dimensions of an array.
- **C = cov(x)** where x is a vector returns the variance of the vector elements.
- **C=xcorr(X,Y)** estimates the cross-correlation sequence of a random process. Autocorrelation is handled as a special case.
- **C=corrcoef(X)** returns a matrix of correlation coefficients calculated from an input matrix whose rows are observations and whose columns are variables.



## Strict-Sense Stationarity (SSS)

- A random process **X(t)** is called **time stationary** or **stationary in the strict sense (SSS)** if all of the distribution functions describing the process are invariant under a time translation, i.e.,

for all $t_1,t_2,\ldots t_k, t_1+\tau, t_2+\tau, \ldots, t_k+\tau \in \Gamma$, and all **k=1,2,…,**

$$P[X(t_1) \le x_1, X(t_2) \le x_2, \ldots, X(t_k) \le x_k]$$
$$= P[X(t_1+\tau) \le x_1, X(t_2+\tau) \le x_2, \ldots, X(t_k+\tau) \le x_k]$$



## Wide-Sense Stationarity (WSS)

- A random process **X(t)** is said to be **stationary in the wide sense (WSS)** if its mean is a constant and the correlation function depends only on the time difference, i.e.,

$$E\{X(t)\} = \mu_X = \text{constant},$$

$$E\{X^*(t)X(t+\tau)\} = R_{XX}(\tau)$$

## Autocorrelation Properties

$R_{XX}(\tau)=E\{X(t)X(t+\tau)\}$

- $R_{XX}(0)=E\{X^2(t)\}\geq 0$ - average power
- $R_{XX}(\tau)$ is an even function of $\tau$, i.e., $R_{XX}(\tau)= R_{XX}(-\tau)$
- $R_{XX}(\tau)$ is bounded by $R_{XX}(0)$ , i.e., $|R_{XX}(\tau)|\leq R_{XX}(0)$
- If $X(t)$ contains a periodic component, then $R_{XX}(\tau)$ will also contain a periodic component
- If $\lim_{\tau\to\infty} R_{XX}(\tau)=C$, then $C= \mu_X^2$
- If $R_{XX}(T_0)= R_{XX}(0)$ for some nonzero $T_0$, then $R_{XX}(\tau)$ is periodic with a period $T_0$
- If $R_{XX}(0)<\infty$, and $R_{XX}(\tau)$ is continuous at $\tau=0$, then it is continuous for every $\tau$



## Frequency Analysis of Random Signals

**How about the spectral properties of the random processes?**

- Direct Fourier transform can not be applied to random signals
- The autocorrelation function $R_{XX}(\tau)$ contains some information about the frequency of the random signals
- Consider real WSS random processes…



## Power Spectral Density Function

- For the random WSS random process $X(t)$, the **PSD function** is defined as

$$S_{XX}(f)=F\{R_{XX}(\tau)\}=\int_{-\infty}^{\infty}R_{XX}(\tau)\exp(-j2\pi f\tau)dt$$

- Given the PSD function, the autocorrelation function can be obtained through the inverse Fourier transform

$$R_{XX}(\tau)=F^{-1}\{S_{XX}(f)\}=\int_{-\infty}^{\infty}S_{XX}(f)\exp(j2\pi f\tau)df$$



## Properties of the PSD Function

The PSD function called the **spectrum of Random process** $X(t)$, has the properties

- $S_{XX}(f)$ is real and nonnegative
- the average power in $X(t)$ is

$$E\{X^2(t)\}=R_{XX}(0)=\int_{-\infty}^{\infty}S_{XX}(f)df$$

- If $X(t)$ is real, then $S_{XX}(f)$ is even
- If $X(t)$ has periodic components, then $S_{XX}(f)$ will have impulses



## Properties of the PSD Function (cont'd)

- Band-related processes
  - Lowpass processes… its psd is zero for $|f|>B$, $B$-bandwidth
  - Bandpass processes… its psd is zero for $f_c-B/2<|f|<f_c+B/2$, $B$-bandwidth, $f_c$ – center frequency
- Power and bandwidth calculations
  - The power within an interval

$$P_X(f_1,f_2)=2\int_{f_1}^{f_2}S_{XX}(f)df$$

  - Effective bandwidth $B_{eff}$, coefficient time $\tau_c$

$$B_{eff}=\frac{1}{2}\frac{\int_{-\infty}^{\infty}S_{XX}(f)df}{\max[S_{XX}(f)]} \qquad \tau_c=\frac{\int_{-\infty}^{\infty}R_{XX}(\tau)d\tau}{R_{XX}(0)}$$



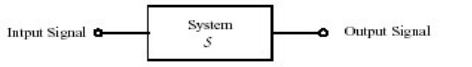
## Ergodicity

- **Motivation**
  - Estimation of the ensemble averages from the time averages …
  - Over a single member function of finite duration…
- **Definition**
  - A stationary random process is **ergodic** if its ensemble averages equal (in a mean-square sense) appropriate time averages
  - Ergodicity is related to specific ensemble averages
- **Benefit**
  - Ergodicity means that any ensemble average of $X(t)$ can be determined from a single member function of $X(t)$ with probability one

## MM1. Response of LTI Systems to Random Inputs

**Reading page: Chapt 4, pp.216-242**

**System:**



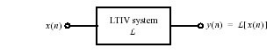

We look at a system as a black box which generates an output signal depending on the input signal and possibly some initial conditions.



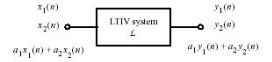
## LTI systems

**1.1. Discrete-time linear time-invariant (LTIV) systems**

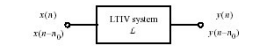
**1.1.1. Discrete-time LTIV system**




*Linear:*




*Time-invariant:*




**1.1.2. Steady-state description of a LTIV system**

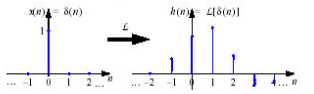
- ***Impulse response:***

The impulse response (IR) $h(n)$ of $\mathcal{L}$ is the response of $\mathcal{L}$ to the unit pulse

$$\delta(n) = \begin{cases} 1\ ; & n = 0 \\ 0\ ; & n \neq 0 \end{cases},$$

namely

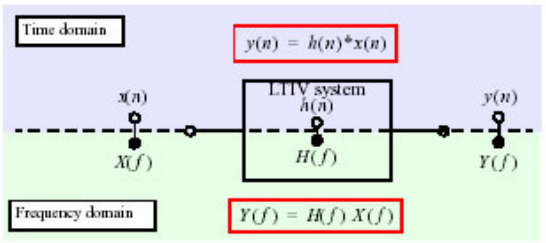
$$h(n) = \mathcal{L}[\delta(n)]$$






## LTI systems (Cont'd)

***Summary: I-O relationship of a LTIV system:***






## Response of LTI Systems to Random Inputs

- LTI system: $y[n]=\Sigma_{k=-\infty}^{\infty} x[k]\, h[n-k]$
- Random inputs
- Two ways for the response computation
  - Compute the response of the LTI system to each member sequence of the random input and then obtain the properties of the ensemble of the output sequences,
  - Computer the properties of the output directly



## Response of Discrete-Time Systems

- A causal LTI system with the IR **h[n]/H(f)**
- The output (**random sequence**) **Y[n]** corresponds the random input sequence **X[n]** is

$$Y[n]=X[n]*h[n]=\Sigma_{k=-\infty}^{\infty} h[k]X[n-k]$$

- The mean sequence of the output **Y[n]** is

$$\mu_Y[n]=E\{Y[n]\}=\Sigma_{k=-\infty}^{\infty} h[k]E\{X[n-k]\}$$

- The autocorrelation sequence of the output **Y[n]** is

$$R_{YY}[n_1,n_2]= \Sigma_{k1=-\infty}^{\infty}\Sigma_{k2=-\infty}^{\infty} h[k_1]h[k_2]\, R_{XX}[n_1-k_1, n_2-k_2]$$



## Response of Discrete-Time Systems (Cont'd)

- If **X[n]** is WSS, then the output will also be WSS, i.e.,

$$\mu_Y=E\{Y[n]\}=\Sigma_{k=-\infty}^{\infty} h[k]E\{X[n-k]\}= \mu_X H(0)$$

- The autocorrelation of **Y[n]**:

$$R_{YY}[n_1,n_2]= \Sigma_{k1=-\infty}^{\infty}\Sigma_{k2=-\infty}^{\infty} h[k_1]h[k_2]R_{XX}[(n_2-n_1)-(k_2-k_1)]$$

$$R_{YY}[n]=R_{XX}[n]* h[-n]* h[n]= R_{hh}[n]* R_{XX}[n]$$

- The psd of **Y[n]** is

$$S_{YY}(f)= S_{XX}(f)H(-f)H(f)=S_{XX}(f)|H(f)|^2$$

## Response of Discrete-Time Systems (Cont'd)

- The cross-correlation between **X[n]** and **Y[n]** is

$$R_{YX}[n]= \Sigma_{k=-\infty}^{\infty} h[-k]R_{XX}[n-k]=h[-n]*R_{XX}[n]$$

$$R_{XY}[n]= \Sigma_{k=-\infty}^{\infty} h[k]R_{XX}[n-k]=h[n]*R_{XX}[n]$$

$$S_{XY}(f)= H(f)S_{XX}(f)$$

**!!!The basis of frequency domain techniques!!! for the design of LTI systems**

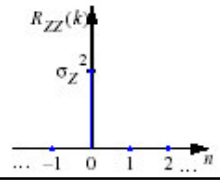


## Special Case: White noise

***White process:***

$Z(n)$ is a white process if it satisfies the following conditions:

- $Z(n)$ is a random process
- $\mu_Z(n) = E[Z(n)] = 0$
- $R_{ZZ}(n, n+k) = E[Z(n)Z(n+k)] = R_{ZZ}(k) = \sigma_Z^2 \delta(k)$



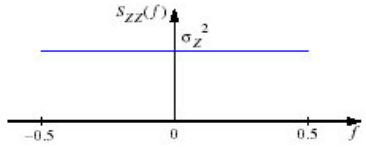



## Special Case: White noise (Cont'd)

***Spectrum of a white process:***

If $Z(n)$ is a white process:

$$S_{ZZ}(f) = \sigma_Z^2$$






## Estimation of System Parameters

- If the input **X[n]** is a white noise, i.e.,

  $\mu_Y= 0$; $R_{XX}[n]= \sigma^2 \delta(n)$, or $S_{XX}(f)=\sigma^2$,

According to

$$R_{XY}[n]= \Sigma_{k=-\infty}^{\infty} h[k]R_{XX}[n-k]=h[n]*R_{XX}[n]$$

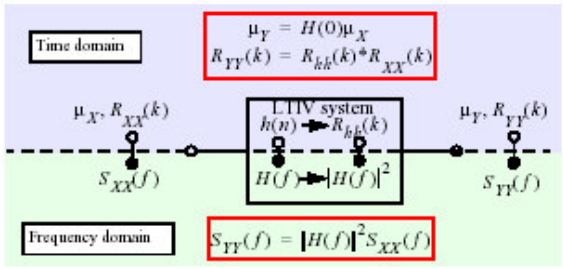
There is

$$R_{XY}[n]= \sigma^2 h[n], \quad \text{or} \quad S_{XY}(f)= \sigma^2 H(f)$$

- It serves as the basis for estimating the impulse/ frequency response of LTI systems if the output in response to a white-noise input can be observable



***Summary: Second-order I-O relationship of a LTIV system:***

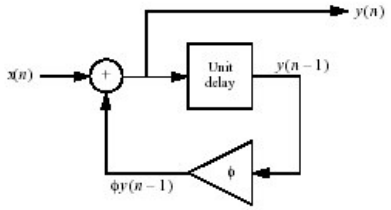





## Example

**1.1.5. Example: First order recursive filter**

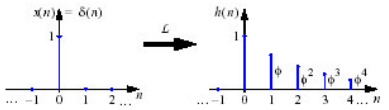
- ***Block diagram and recursive equation:***




$$y(n) = x(n) + \phi y(n-1) \qquad (y(n) = 0 \quad n < 0)$$

## Example (Cont'd)

- ***Impulse response:***




$$h(n) = \begin{cases} 0 ; & n < 0 \\ \phi^n ; & n \geq 0 \end{cases}$$

- ***Stability condition:***

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |\phi|^n = \lim_{N\to\infty} \frac{1-|\phi|^N}{1-|\phi|} < \infty \quad \Leftrightarrow \quad |\phi| < 1$$

$$\left[\sum_{n=0}^{N} a^n = \frac{1-a^{N+1}}{1-a}\right]$$



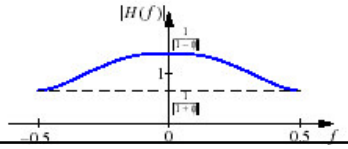
## Example (Cont'd)

- ***Transfer function:***

$$H(f) = \mathcal{F}\{h(n)\} = \sum_{n=0}^{\infty} \phi^n \exp(-j2\pi nf)$$

$$= \sum_{n=0}^{\infty} [\phi \exp(-j2\pi f)]^n$$

$$= \frac{1}{1-\phi\exp(-j2\pi f)}$$






## Example (Cont'd)

- ***Second-order I-O relationship:***

  - Time domain:

$$\mu_Y = H(0)\mu_X = \frac{1}{1-\phi}\mu_X$$

$$R_{YY}(k) = R_{hh}(k) * R_{XX}(k) = \frac{\phi^{|k|}}{1-\phi^2} * R_{XX}(k)$$

$$\left[R_{hh}(k) = \sum_{m=0}^{\infty} \phi^m \phi^{m+|k|} = \phi^{|k|} \sum_{m=0}^{\infty} \phi^{2m} = \phi^{|k|} \frac{1}{1-\phi^2}\right]$$

  - Frequency domain:

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f) = \frac{1}{|1-\phi\exp(-j2\pi f)|^2} S_{XX}(f)$$



## Example (Cont'd)

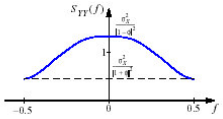
- ***Special case: AR(1) process (see Section 2.2):***

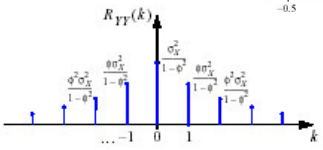
If $X(n)$ is a white Gaussian process,

$$\mu_Y = 0$$

$$S_{YY}(f) = \frac{\sigma_X^2}{|1-\phi\exp(-j2\pi f)|^2}$$

$$R_{YY}(k) = \frac{\phi^{|k|}}{1-\phi^2}\sigma_X^2$$






## Response of Continuous-Time Systems

- Deterministic case
  - Output of a causal LTI system is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

  - Frequency expression $\quad Y(f) = X(f)H(f)$

- The output (**random process**) **Y(t)** of a causal LTI system in response to a random input process **X(t)** is

$$Y(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$$



## Response of Continuous-Time Systems (Cont'd)

- The mean function of the output **Y(t)** is

$$\mu_Y(t) = E\{Y(t)\} = \int_{-\infty}^{\infty} \mu_X(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\mu_X(t-\tau)d\tau$$

- The autocorrelation function of the output **Y(t)** is

$$R_{YY}(t_1,t_2) = E\{Y(t_1)Y(t_2)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(t_1-\tau_1, t_2-\tau_2)d\tau_1 d\tau_2$$

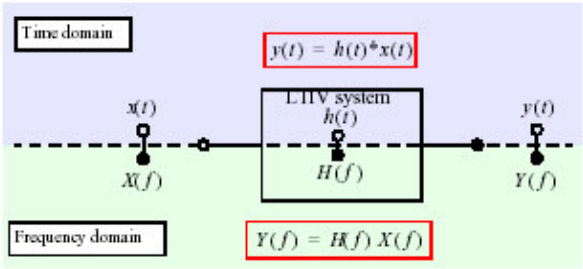
## Response of Continuous-Time Systems (Cont'd)

- The cross-correlation between **X(t)** and **Y(t)** is
  $R_{YX}(\tau)= h(-\tau)*R_{XX}(\tau)$, and $R_{XY}(\tau)= h(\tau)*R_{XX}(\tau)$
- The autocorrelation of **Y(t)** is
  $R_{YY}(\tau)=R_{YX}(\tau)*h(\tau)= R_{XX}(\tau)* h(-\tau)* h(\tau)$
- The psd function of **Y[n]** is
  $S_{YY}(f)= S_{XX}(f)H(-f)H(f)=S_{XX}(f)|H(f)|^2$

**$|H(f)|^2$ is referred to as the power transfer function**

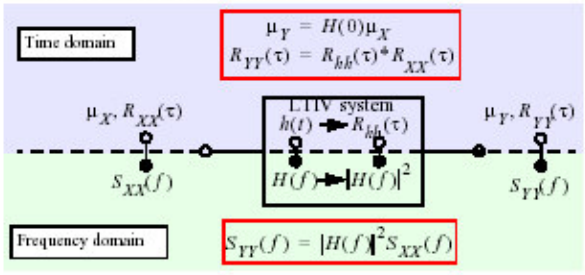


- ***Summary: I-O relation ship of a LTIV system:***






- ***Summary: Second order I-O relationship of a LTIV system:***

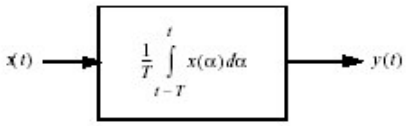





## Example

**1.2.5. Example: Ideal integrator**

- ***Block diagram and input-output relationship:***




$$y(t) = \frac{1}{T}\int_{t-T}^{t} x(\alpha)d\alpha$$