# MM3. Signal Detection (Part One)

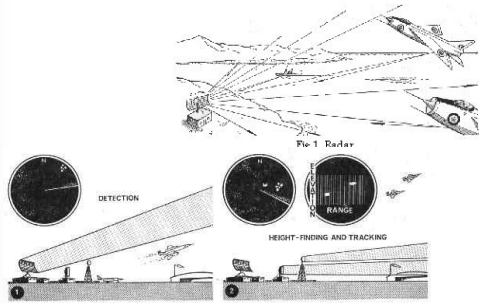
- Hypothesis testing
- Decision rules
- Binary detection

**Reading page: Chapt 6, pp.341-352**



## Motivation

- Target detection and tracking – Radar system






## Motivation (Cont'd)

- Fault detection

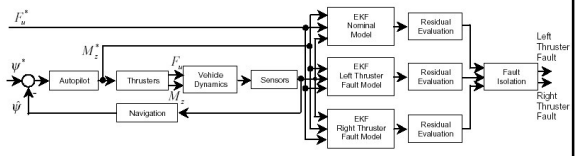



**Figure 1** An observer-based fault diagnostic scheme (Allesandri *et al*, 1999a).

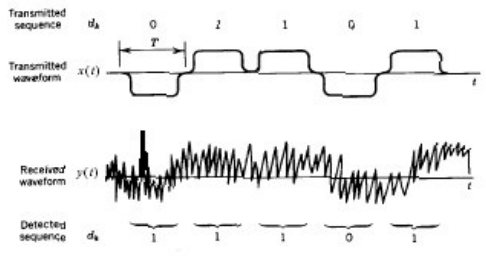


## Signal Detection and Estimation

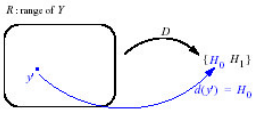
***Example 1:*** *Detection of a BPAM signal:*

$H_0$ : "0" is transmitted  Null hypothesis

$H_1$ : "1" is transmitted  Alternate hypothesis





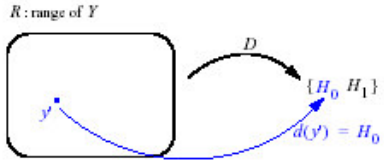

## 3.1 Hypothesis Testing

In hypothesis testing, a **decision** is made based on the **observation** of a random variable as to which of several **hypotheses** to accept



## Binary Hypothesis Testing

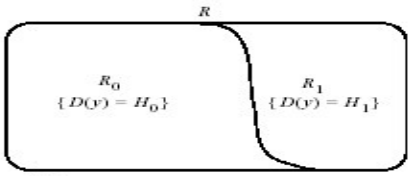
- Two hypotheses: $H_0$ and $H_1$
- One observation y of a random variable Y whose pdf under each hypothesis is known, denoted as $f(y|H_0)$ and $f(y|H_1)$
- A decision rule is to decide
- $D: R \rightarrow \{H_0, H_1\}$,  R range of Y



## Binary Hypothesis Testing (Cont'd)

The decision rule $D$ determines two **decision regions** in $R$:

$$R_0 : y \in R_0 \quad \Rightarrow \quad D(y) = H_0$$
$$R_1 : y \in R_1 \quad \Rightarrow \quad D(y) = H_1$$




Properties of $R_0$ and $R_1$:

- $R_0 \cup R_1 = R$
- $R_0 \cap R_1 = \varnothing$



## Decision Rule

- ***Decision table:***

| Decision $D$ | True hypothesis $H$ | |
|---|---|---|
| | $H_0$ | $H_1$ |
| $H_0$ | $(H_0, H_0)$ | $(H_0, H_1)$ |
| $H_1$ | $(H_1, H_0)$ | $(H_1, H_1)$ |

- : Correct decision
- : Incorrect decision:

The pair $(H_i, H_j)$ $(i, j = 0, 1)$ in the above table means $D = H_i$ and $H = H_j$.



## Probabilities of Decisions

- ***Probabilities of correct decision and of making an error:***
  - Probability of correct decision:

$$P_c = P[D = H_0, H = H_0] + P[D = H_1, H = H_1]$$
$$= P[D = H_0 | H_0]P[H_0] + P[D = H_1 | H_1]P[H_1]$$

  - Probability of incorrect decision:

$$P_e = P[D = H_1, H = H_0] + P[D = H_0, H = H_1]$$
$$= P[D = H_1 | H_0]P[H_0] + P[D = H_0 | H_1]P[H_1]$$

Obviously,

$$P_c = 1 - P_e.$$



## Type I and II Errors

- ***Types of error and their probability:***
  - **False alarm** (Type I error): $D = H_1$ when $H_0$ is true.
    False alarm probability:

$$P_f = P[D = H_1 | H_0].$$

  - **Miss** (Type II error): $D = H_0$ when $H_1$ is true.
    Probability of a miss:

$$P_m = P[D = H_0 | H_1].$$

$P_e = P_f P(H_0) + P_m P(H)$



## Example: BPAM Signal Detection

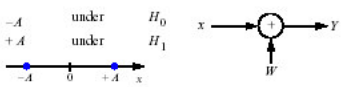
***Example:*** Binary pulse amplitude modulation (BPAM)

- Signal model:

$$Y = x + W$$

- where

$$x = \begin{cases} -A & \text{under } H_0 \\ +A & \text{under } H_1 \end{cases}$$




- $W$ is a Gaussian noise, i.e.:
  - $W$ is a Gaussian random variable,
  - with expectation $\mathbf{E}[W] = 0$,
  - and variance $\mathbf{E}[W^2] = \sigma^2$.

Probability density function (pdf) of $W$:

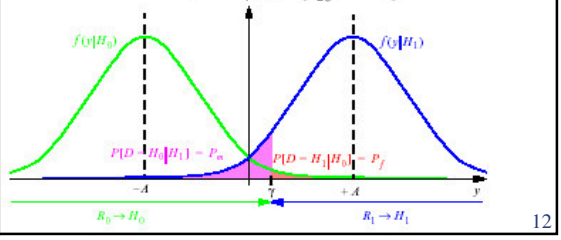
$$f(w) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2} w^2\right\}$$



## Example: BPAM Detection (Cont'd)

- Pdf of $Y$ under $H_0$ and $H_1$:

$$f(y|H_0) = f(w)\big|_{w=y+A} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(y+A)^2\right\}$$
$$f(y|H_1) = f(w)\big|_{w=y-A} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(y-A)^2\right\}$$



## 3.2 Decision Rules

- **Maximum "a posteriori" decision rule**
- Bayes decision rule – Costs of errors
- Minmax rule and Neyman-Pearson rule



## Maximum "a posteriori" (MAP) Rule

- The MAP rule is of the form:

$$P(H_1 \mid y) \underset{H_0}{\overset{H_1}{\gtrless}} P(H_0 \mid y)$$

- $P(H_i|y)$, i=1, 2, is the "a posteriori" probability of $H_i$ when $Y=y$ is observed
- Bayes rule:

$$P(H_i \mid y) = \frac{f(y \mid H_i)P(H_i)}{f(y)} \quad \Leftrightarrow \quad P(H_i \mid y) = \frac{P(y \mid H_i)P(H_i)}{P(y)}$$



## MAP Rule (Cont'd) – Likelihood Ratio

- MAP decision rule:

$$f(y \mid H_1)P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} f(y \mid H_0)P(H_0)$$

- Prior distribution $P(H_i)$, i=1,2
- Likelihood ratio L(y):

$$L(y) = \frac{f(y \mid H_1)}{f(y \mid H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

$$l(y) = \ln(L(y))$$

$$l(y) = \ln\left(\frac{f(y \mid H_1)}{f(y \mid H_0)}\right) \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P(H_0)}{P(H_1)}\right)$$

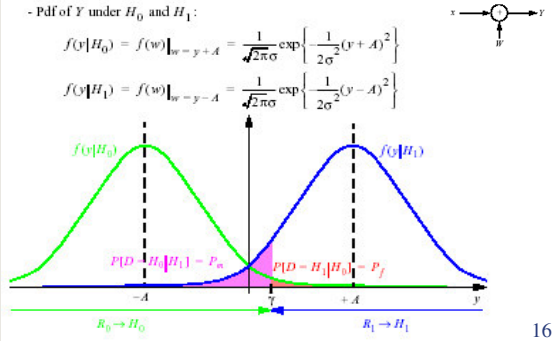


## Example: BPAM MAP Detection

- Pdf of $Y$ under $H_0$ and $H_1$:

$$f(y|H_0) = f(w)\big|_{w=y+A} = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}(y+A)^2\right\}$$

$$f(y|H_1) = f(w)\big|_{w=y-A} = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{1}{2\sigma^2}(y-A)^2\right\}$$






## Example: BPAM MAP Detection

- Likelihood ratio:

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} = \frac{\exp\left\{-\frac{1}{2\sigma^2}(y-A)^2\right\}}{\exp\left\{-\frac{1}{2\sigma^2}(y+A)^2\right\}} = \exp\left\{\frac{1}{2\sigma^2}\left[(y+A)^2-(y-A)^2\right]\right\}$$

- Loglikelihood ratio:

$$l(y) = \ln\left(\frac{f(y|H_1)}{f(y|H_0)}\right) = \frac{1}{2\sigma^2}\left[(y+A)^2-(y-A)^2\right] = \frac{2Ay}{\sigma^2}$$

- MAP decision rule:

$$\frac{2Ay}{\sigma^2} \underset{H_0}{\overset{H_1}{\gtrless}} \ln\left(\frac{P[H_0]}{P[H_1]}\right)$$

$$y \underset{H_0}{\overset{H_1}{\gtrless}} \frac{\sigma^2}{2A}\ln\left(\frac{P[H_0]}{P[H_1]}\right) = \hat{\gamma}_{MAP}$$

***Maximum likelihood (ML) decision rule:***

Selecting a uniform "a priori" pdf for the hypotheses, i.e.

$$P[H_0] = P[H_1] = \frac{1}{2},$$

the MAP decision rule reduces to the ML decision rule:

$$f(y|H_1) \underset{H_0}{\overset{H_1}{\gtrless}} f(y|H_0)$$



## 3.2 Decision Rules

- Maximum "a posteriori" decision rule
- **Bayes decision rule – Costs of errors**
- Minmax rule and Neyman-Pearson rule

## Bayes' Decision Rule – Cost of Errors

- Motivation: cost of wrong decisions

***Cost function:***
In many engineering branches costs have to be taken into account depending on the decision and the true hypothesis.

| Decision $D$ | True hypothesis | |
|---|---|---|
| | $H_0$ | $H_1$ |
| $H_0$ | $C_{00}$ | $C_{01}$ |
| $H_1$ | $C_{10}$ | $C_{11}$ |

Usually, the cost of making a wrong decision is higher than that of making a correct decision:

$$C_{10} \geq C_{00} \quad \text{and} \quad C_{01} \geq C_{11}.$$



## Bayes' Decision Rule

- Average cost:

$$\bar{C} = C_{00}P[D=H_0|H_0]P[H_0] + C_{10}P[D=H_1|H_0]P[H_0] + C_{01}P[D=H_0|H_1]P[H_1] + C_{11}P[D=H_1|H_1]P[H_1]$$

- Bayes' decision rule: minimize the average cost

$$L(y) = \frac{f(y|H_1)}{f(y|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)(C_{10}-C_{00})}{P(H_1)(C_{01}-C_{11})}$$



## Bayes' Decision Rule – How?

***Proof:***
The average cost $\bar{C}$ can be written as:

$$\bar{C} = C_{10}P[H_0] + C_{11}P[H_1] + \int_{R_0} \{P[H_1](C_{01}-C_{11})f(y|H_1) - P[H_0](C_{10}-C_{00})f(y|H_0)\}dy$$

□

Note that the Bayes rule reduces to the MAP rule when the cost is selected to be:

$$C_{00} = C_{11} = 0$$
$$C_{10} = C_{01} = 1$$

See page349-350



## 3.2 Decision Rules

- Maximum "a posteriori" decision rule
- Bayes decision rule – Costs of errors
- **Minmax rule and Neyman-Pearson rule**



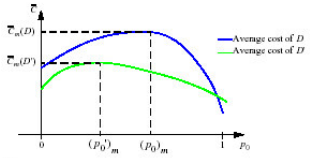
## Minmax Decision Rule

- **Motivation**

  The priori distribution $P(H_i)$, i=1,2 is unknown

- Maximum average cost of a decision rule:

Let us consider the behavior of $\bar{C}$ for a fixed decision rule $D$ as $p_0 = P[H_0]$ varies:



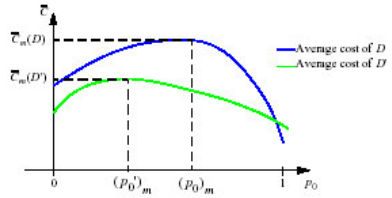

$\bar{C}_{m'}(D)$ is the maximum average cost and $P[H_0] = (p_0)_m$, $P[H_1] = 1-(p_0)_m$ is the worst case "a priori" pdf when employing decision rule $D$.



## Minmax Decision Rule (Cont'd)

- Minimax decision rule: minimize the maximum average cost of all possible decision rules:

$$\bar{C}_m(D_{MM}) \leq \bar{C}_m(D) \quad for \quad any \quad decision \quad rule \quad D$$



# Neyman-Pearson Decision Rule

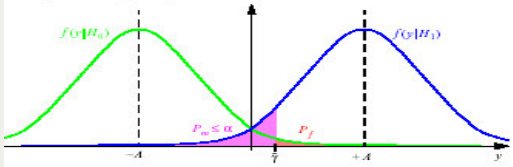
The Neyman-Pearson (NP) decision rule is used when neither an "a priori" pdf nor cost assignments are given.

A NP decision rule minimizes the probability of false alarm

$P_f = P[D = H_1 | H_0]$ while keeping the probability of a miss

$P_m = P[D = H_0 | H_1]$ below a certain specified level, say $\alpha$.

***Example:*** BPAM (cont'd)




Thus, a NP decision rule $D_{NP_\alpha}$ satisfies the inequality

$$P_f(D_{NP_\alpha}) \le P_f(D) \quad \text{for any decision rule } D \text{ such that } P_m(D) \le \alpha$$