MM6. Discrete-Time Wiener Filters

Reading page: Chapt 7, pp.406-419

- Explain MM5 exercise
- 6.1 (Ideal) Noncausal Wiener Filters
- 6.2 Causal Wiener Filters

What have we talked through MM5 – **Minimum Mean Squared Error Estimation**?

- 5.1 Linear minimum mean squared error estimators
- 5.2 (Nonlinear) minimum mean squared error estimators

5.1 Linear Minimum Mean Squared Error (LMMSE) Estimators

LMMSE Problem formulation

- A random sequence X(1), ... X(M) whose realization can be observed
- A random variable Y which has to be estimated
- Seek a linear estimator as:

$$\hat{Y} = h_0 + \sum_{m=1}^{M} h_m X(m)$$

- By miminizing the mean squared error(MSE):

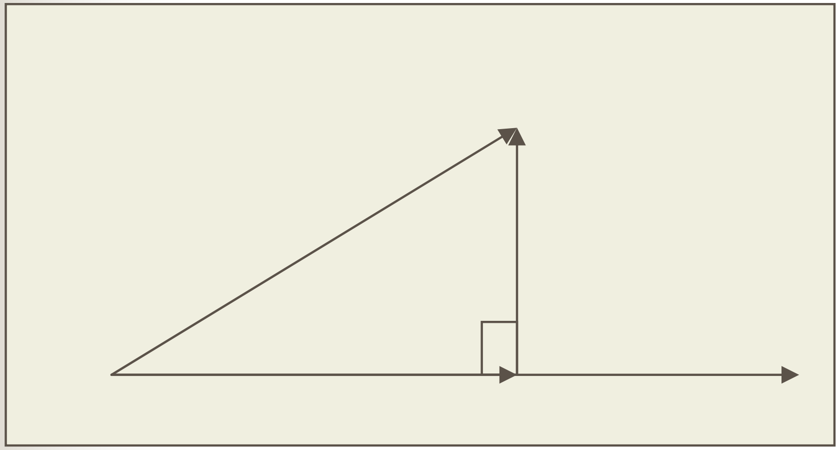
$$\min_{h_m, m=0,1,\ldots M} E\{(Y-\hat{Y})^2\}$$

5.1.1 Orthogonality Principle

- A necessary condition for a linear estimator denoted by $\mathbf{h}=[h_0, \ldots h_M]^T$ to be the solution of the LMMSE is that

$$E\{Y-\hat{Y}\} = E\left\{Y-\left(h_0+\sum_{m=1}^{M} h_m X(m)\right)\right\} = 0 \qquad E\{Y\} = E\{\hat{Y}\}$$

$$E\left\{(Y-\hat{Y})X(j)\right\} = E\left\{\left(Y-\left(h_0+\sum_{m=1}^{M} h_m X(m)\right)\right)X(j)\right\} = 0, \quad j=1,\ldots,M$$



- Orthogonality Principle

5.1.3 LMMSE Soultion

$$\mathbf{h} = [h_1 \quad \cdots \quad h_M]^T \qquad \mu_{\mathbf{X}} = [\mu_{X(1)} \quad \cdots \quad \mu_{X(M)}]^T$$

$$\Sigma_{\mathbf{X}Y} = \begin{bmatrix} \Sigma_{X(1)Y} \\ \vdots \\ \Sigma_{X(M)Y} \end{bmatrix}, \qquad \Sigma_{\mathbf{XX}} = \begin{bmatrix} \Sigma_{X(1)X(1)} & \cdots & \Sigma_{X(1)X(M)} \\ \vdots & \ddots & \vdots \\ \Sigma_{X(M)X(1)} & \cdots & \Sigma_{X(M)X(M)} \end{bmatrix},$$

$$\mathbf{h} = \left(\Sigma_{\mathbf{XX}}\right)^{-1} \Sigma_{\mathbf{X}Y}$$

$$h_0 = \mu_Y - \mathbf{h}^T \mu_{\mathbf{X}} = \mu_Y - \left(\Sigma_{\mathbf{X}Y}\right)^T \left(\Sigma_{\mathbf{XX}}\right)^{-1} \mu_{\mathbf{X}}$$

$$E\{(Y-\hat{Y})^2\} = E\{Y^2\} - E\{\hat{Y}^2\} = E\{(Y-\hat{Y})Y\}$$

$$E\{(Y-\hat{Y})^2\} = \sigma_Y{}^2 - \mathbf{h}^T \Sigma_{\mathbf{X}Y}$$

5.2 Minimum Mean Squared Error Estimators (MMSEE)

- **MMSEE:** The solution for MMSEE of Y based on the observation of X(1),....,X(M) is:

$$\tilde{Y}(X(1),\ldots,X(M)) = E\{Y \mid X(1),\ldots,X(M)\}$$

- Which reaches $\min E\{(Y-\tilde{Y})^2\}$
- Specially, if X(1)=x(1),....,X(M)=x(M) is observaed, then

$$\begin{aligned}\tilde{Y}(x(1),\ldots,x(M)) &= E\{Y \mid x(1),\ldots,x(M)\} \\ &= \int y p(y \mid x(1),\ldots,x(M)) dy\end{aligned}$$

- $\tilde{Y} = \hat{Y}$ iff [Y X(1),....,X(M)] is a Gaussian vector

Explain the MM5 Exercise!

MM6. Discrete-Time Wiener Filters

Reading page: Chapt 7, pp.406-419

- 6.1 **Noncausal Wiener Filters**
- 6.2 Causal Wiener Filters

Discrete-Time Wiener Filters

Motivation:

- Estimate a WSS random sequence Y(n) based on the observation of another sequence X(n).

- **Without loss of generality we assume that**

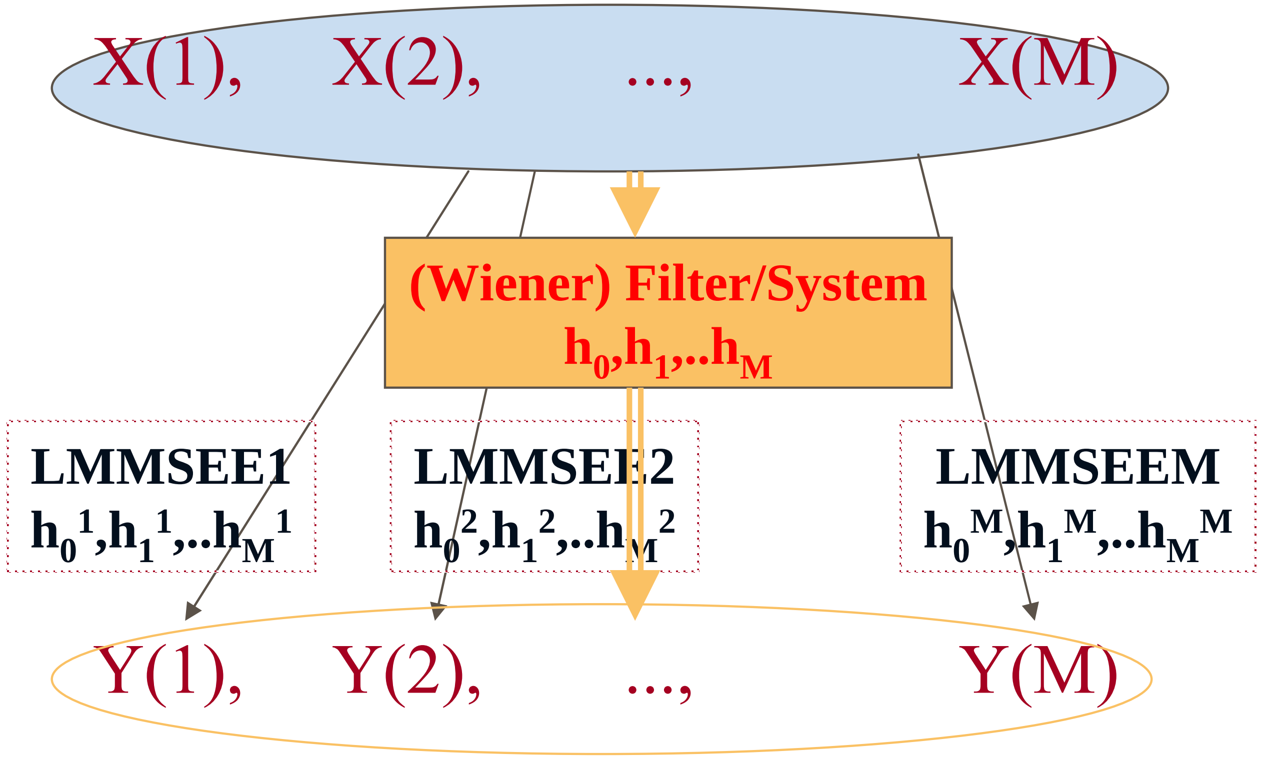
$$\mathbf{E\{Y(n)\}=E\{X(n)\}=0}$$

- The goodness of the estimator is described by MSE

$$E\{(Y(n)-\hat{Y}(n))^2\}$$

6.0 An Intuitive Explanation

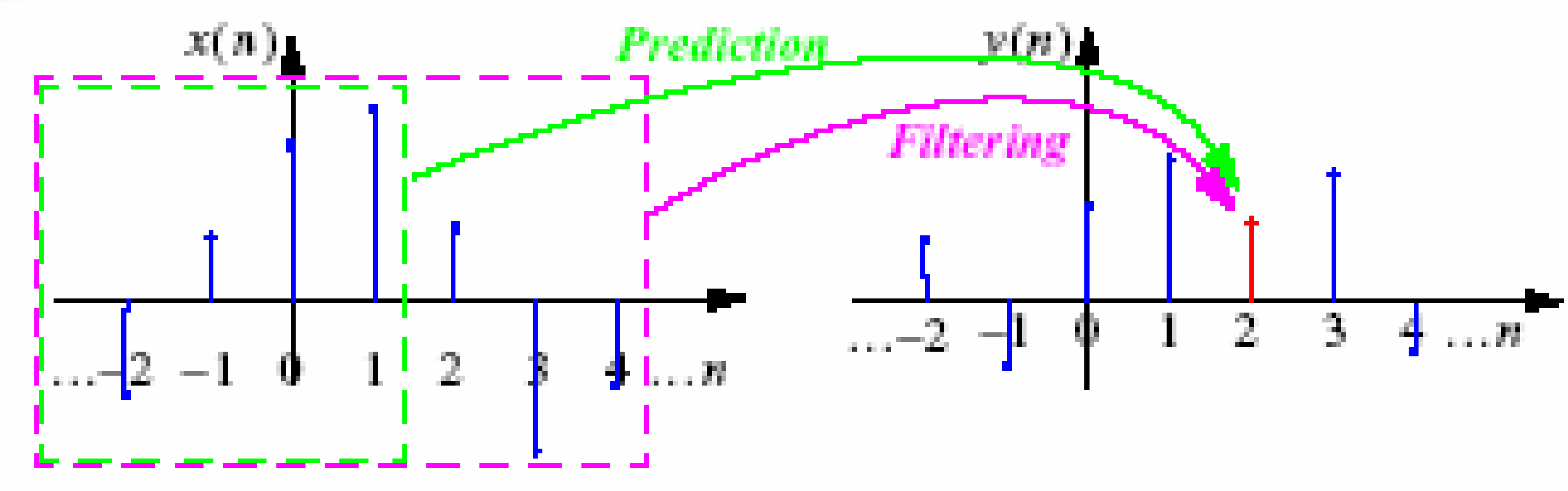
- Observable random sequence:



- A random sequence which needs to be estimated

6.0 Prediction and Filtering

- **Prediction:** $\hat{Y}(n)$ depends on past observations of $X(n)$
- **Filtering:** $\hat{Y}(n)$ depends on the present observation and/or one or many future observations $X(n)$
- **Causal, noncausal filters**



6.1 Ideal (Noncausal) Wiener Filters

- **Problem Formulation:**

Seek a linear filter ← **system**

$$\hat{Y}(n) = \sum_{m=-\infty}^{\infty} h(m) X(n-m) = h(n) * X(n)$$

$$LMMSEE: \quad \hat{Y} = h_0 + \sum_{m=1}^{M} h_m X(m)$$

↑ Signals' relationship

Which minimizes the MSE

$$E\{(\hat{Y}(n) - Y(n))^2\}$$

The filter reaching above requirement is called *ideal (noncausal) Wiener filter*

6.1.1 Orthogonal Principle

- The coefficients of a Wiener filter satisfy

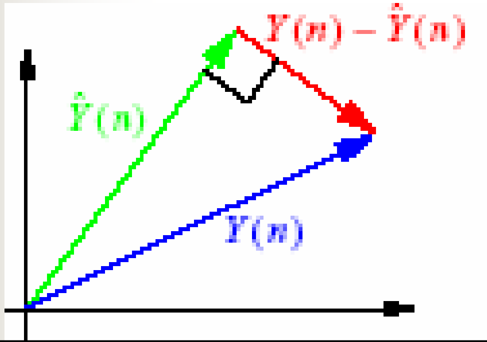
$$E\{(Y(n)-\hat{Y}(n))X(n-k)\} = E\{(Y(n)-\sum_{m=-\infty}^{\infty} h(m)X(n-m))X(n-k)\} = 0$$

- Consequences:

$$E\{(Y(n)-\hat{Y}(n))\hat{Y}(n)\} = 0$$

$$E\{(Y(n)-\hat{Y}(n))^2\} = E\{(Y(n))^2\} - E\{(\hat{Y}(n))^2\}$$

$$= E\{(Y(n)-\hat{Y}(n))Y(n)\}$$



6.1.1 Orthogonal Principle (Cont'd)

- Define the crosscorrelation and autocorrelation:

$$R_{XY}(k) = E\{(X(n)Y(n+k)\}, \quad R_{XX}(k) = E\{(X(n)X(n+k)\}$$

- Consequence of orthogonal principle: (Wiener-Hopf equation)

$$E\{(Y(n) - \hat{Y}(n))X(n-k)\} = E\{[Y(n) - \sum_{m=-\infty}^{\infty} h(m)X(n-m)]X(n-k)\} = 0$$

$$R_{XY}(k) = \sum_{m=-\infty}^{\infty} h(m)R_{XX}(k-m) = h(k) * R_{XX}(k)$$

6.1.2 Wiener Filter in Transfer Function

Wiener-Hopf equation:

$$R_{XY}(k) = \sum_{m=-\infty}^{\infty} h(m) R_{XX}(k-m) = h(k) * R_{XX}(k)$$

- TF of the Wiener filter:

$$'H(f) = \frac{S_{XY}(f)}{S_{XX}(f)}$$

- MSE residual:

$$E\{(Y(n) - \hat{Y}(n))^2\} = \sigma_Y{}^2 - \sum_{m=-\infty}^{\infty} h(m) R_{XY}(m)$$

$$E\{(Y(n) - \hat{Y}(n))^2\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} [S_{YY}(f)] - \frac{|S_{XY}(f)|2}{S_{XX}(f)}]df$$

See p.408-409 for proof

6.1.3 Wiener Filter via LMMSEE

- Wiener filter:

$$R_{XY}(k) = \sum_{m=-\infty}^{\infty} h(m) R_{XX}(k-m) = h(k) * R_{XX}(k)$$

$$'H(f) = \frac{S_{XY}(f)}{S_{XX}(f)}$$

- MSE residual:

$$E\{(Y(n) - \hat{Y}(n))^2\} = \sigma_Y{}^2 - \sum_{m=-\infty}^{\infty} h(m) R_{XY}(m)$$

$$E\{(Y(n) - \hat{Y}(n))^2\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} [S_{YY}(f)] - \frac{|S_{XY}(f)|2}{S_{XX}(f)}]df$$

- LMMSEE:

$$\mathbf{h} = [h_1 \quad \cdots \quad h_M]^T \qquad \mu_{\mathbf{X}} = [\mu_{\mathbf{X}(1)} \quad \cdots \quad \mu_{\mathbf{X}(M)}]^T$$

$$\sum_{\mathbf{X}Y} = \begin{bmatrix} \sum_{X(1)Y} \\ \vdots \\ \sum_{X(M)Y} \end{bmatrix}, \qquad \sum_{\mathbf{XX}} = \begin{bmatrix} \sum_{X(1)X(1)} & \cdots & \sum_{X(1)X(M)} \\ \vdots & \ddots & \vdots \\ \sum_{X(M)X(1)} & \cdots & \sum_{X(M)X(M)} \end{bmatrix},$$

$$\mathbf{h} = \left(\sum_{\mathbf{XX}}\right)^{-1} \sum_{\mathbf{X}Y}$$

$$h_0 = \mu_Y - \mathbf{h}^T \mu_{\mathbf{X}} = \mu_Y - \left(\sum_{\mathbf{X}Y}\right)^T \left(\sum_{\mathbf{XX}}\right)^{-1} \mu_{\mathbf{X}}$$

- MSE residual:

$$E\{(Y - \hat{Y})^2\} = E\{Y^2\} - E\{\hat{Y}^2\} = E\{(Y - \hat{Y})Y\}$$

$$E\{(Y - \hat{Y})^2\} = \sigma_Y{}^2 - \mathbf{h}^T \sum_{\mathbf{X}Y}$$

MM6. Discrete-Time Wiener Filters

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- 6.1 Noncausal Wiener Filters
- 6.2 **Causal Wiener Filters**

6.2.1 Causal Wiener Filter (Case A)

Case A: X(n) is a white noise with unity variance

$E\{X(n)X(n+k)\}=\delta(k)$

- Ideal Wiener filter:

$$\hat{Y}(n)=\sum_{m=-\infty}^{\infty}h(m)X(n-m)=h(n)*X(n)$$

- **A causal Wiener filter** can be achieved by cancelling the noncausal part of ideal Wiener filter:

$$\hat{Y}_c(n)=\sum_{m=0}^{\infty}h(m)X(n-m)$$

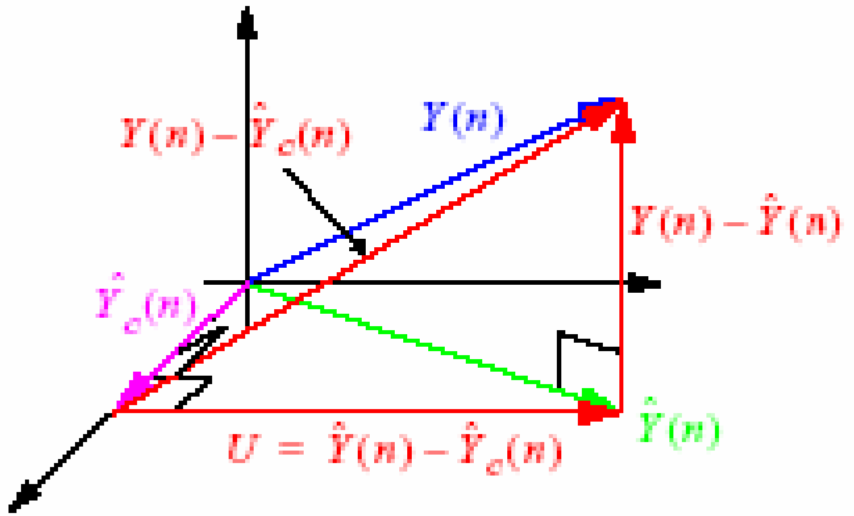
- The causal Wiener filter minimizes the MSE within the class of causal linear estimators

Sketch of the proof:

$\hat{Y}(n)$ can be written as

$$\hat{Y}(n) = \underbrace{\sum_{m=-\infty}^{-1} h(m)X(n-m)}_{\equiv U} + \underbrace{\sum_{m=0}^{\infty} h(m)X(n-m)}_{V \equiv \hat{Y}_c(n)}$$

Because $X(n)$ is a white noise, the causal part $V = \hat{Y}_c(n)$ and the noncausal part $U = \hat{Y}(n) - \hat{Y}_c(n)$ of $\hat{Y}(n)$ are orthogonal. It follows from this property



that $\hat{Y}_c(n)$ and $Y(n)$ are orthogonal, i.e. that $\hat{Y}_c(n)$ minimizes the MSE within the class of linear causal estimators.

□

6.2.2 Causal Wiener Filter (Case B)

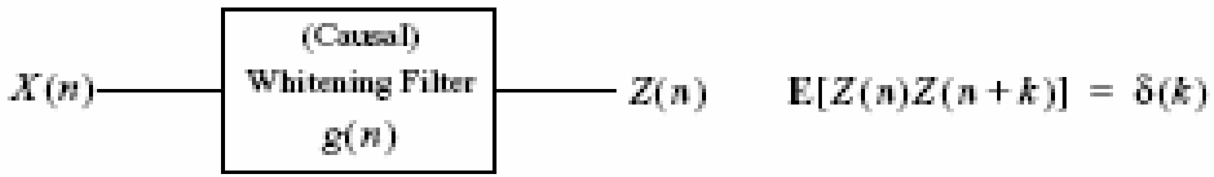
Case B: X(n) is a WSS signal

- **Causal whitening filter:**

If $S_{XX}(f)$ satisifies the Paley-Wiener condition:

$$\int_{-1/2}^{1/2} \log(S_{XX}(f))df > -\infty$$

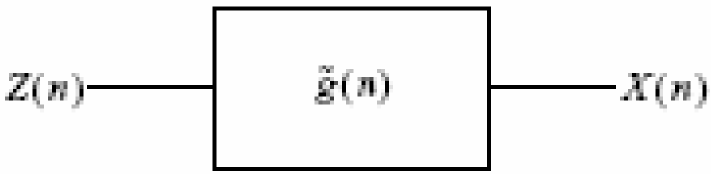
then X(n) can be converted into an equivalent white noise sequence Z(n) with unit variance by filtering it with an appropriate causal filter g(n)



6.2.3 Whitening and Whitening Filter

This operation is called whitening and the filter $g(n)$ is called a whitening filter.

equivalent $\equiv$ there exists another causal filter $\tilde{g}(n)$ so that $X(n) = \tilde{g}(n) * Z(n)$:



Notice that if

$$G(f) = \mathcal{F}\{g(n)\}$$

$$\tilde{G}(f) = \mathcal{F}\{\tilde{g}(n)\}$$

then

$$|G(f)|^2 = S_{XX}(f)^{-1}$$

$$|\tilde{G}(f)|^2 = S_{XX}(f)$$

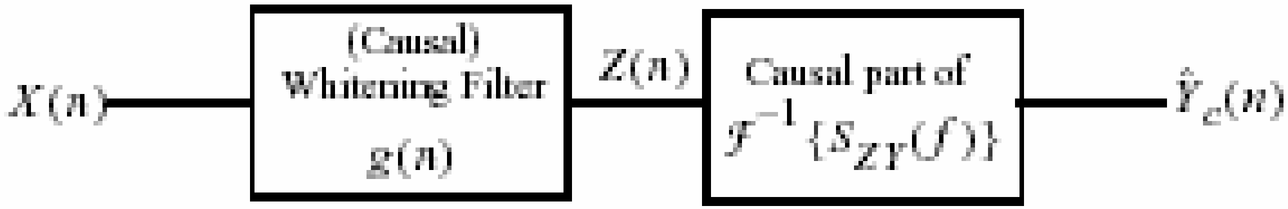
We shall see that a whitening filter exists such that

$$\tilde{G}(f) = G(f)^{-1} \qquad (3.6)$$

6.2.4 Causal Wiener Filter (Whitening)

- ***Causal Wiener filter***

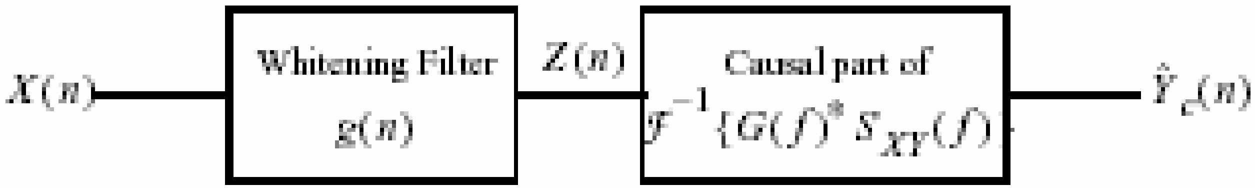
  Making use of the result in Part A, the block diagram of the causal Wiener filter is



$S_{ZY}(f)$ is obtained from $S_{XY}(f)$ according to

$$S_{ZY}(f) = G(f)^* S_{XY}(f)$$

Hence, the block diagram of the causal Wiener filter is:



6.2.5 Spectral Decomposition Theorem

- ***Spectral Decomposition Theorem:***

  Let $S_{XX}(f)$ satisfies the so-called **Paley-Wiener condition**:

$$\int_{-1/2}^{1/2} \log(S_{XX}(f))df > -\infty$$

Then $S_{XX}(f)$ can be written as

$$S_{XX}(f) = G(f)^{+} G(f)^{-}$$

with $G(f)^{+}$ and $G(f)^{-}$ satisfying

$$\left|G(f)^{+}\right|^2 = \left|G(f)^{-}\right|^2 = S_{XX}(f).$$

Moreover, the sequences

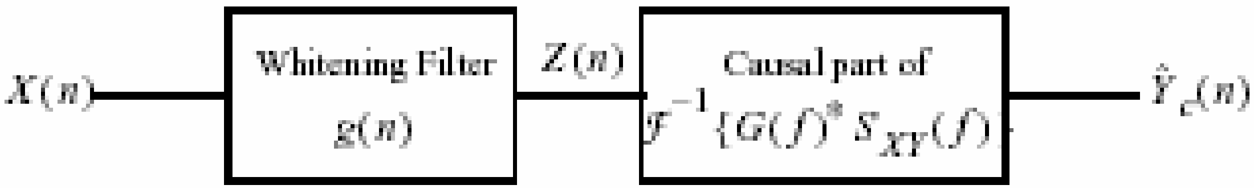
$$g(n)^{+} = \mathcal{F}^{-1}\{G(f)^{+}\}$$
$$g(n)^{-} = \mathcal{F}^{-1}\{G(f)^{-}\}$$
$$g^{-1}(n)^{+} = \mathcal{F}^{-1}\{1/G(f)^{+}\}$$
$$g^{-1}(n)^{-} = \mathcal{F}^{-1}\{1/G(f)^{-}\}$$

satisfy

$g(n)^{+} = g^{-1}(n)^{+} = 0$	$n < 0$	Causal sequences
$g(n)^{-} = g^{-1}(n)^{-} = 0$	$n > 0$	Anticausal sequences

6.2.6 Causal Wiener Filter (Cont'd)

Hence, the block diagram of the causal Wiener filter is:



The sought whitening filter used to obtained $Z(n)$ is

$$g(n) = g^{-1}(n)^{+}$$

and

$$\tilde{g}(n) = g(n)^{+}.$$

It can be easily verified that both sequences satisfy the identities in (3.6).

6.2.7 Finite Wiener Filter

$$Finite\ Wiener\ filter: \quad \hat{Y}(n) = \sum_{m=0}^{M} h(m) X(n-m)$$

$$Wiener-Hopf\ Solution:$$

$$\mathbf{h}^{\mathbf{T}} = [h(0)\ h(1) \cdots h(M)] = \mathbf{R_{XX}}^{-1}\mathbf{R_{XY}}$$

provided $\mathbf{R_{XX}}$ *is invertible,*

where

$$\mathbf{R_{XX}} = \begin{bmatrix} R_{XX}(0) & R_{XX}(1) & \cdots & R_{XX}(M) \\ R_{XX}(1) & R_{XX}(0) & \cdots & R_{XX}(M-1) \\ \vdots & \vdots & \ddots & \vdots \\ R_{XX}(M) & R_{XX}(M-1) & \cdots & R_{XX}(0) \end{bmatrix}$$

$$\mathbf{R_{XY}}^{\mathbf{T}} = [R_{XY}(0)\ R_{XY}(1) \cdots R_{XY}(M)]$$

- LMMSEE:

$$\mathbf{h} = [h_1 \ \cdots \ h_M]^T \qquad \mu_{\mathbf{X}} = [\mu_{\mathbf{X}(1)} \ \cdots \ \mu_{\mathbf{X}(M)}]^T$$

$$\sum_{XY} = \begin{bmatrix} \sum_{X(1)Y} \\ \vdots \\ \sum_{X(M)Y} \end{bmatrix}, \qquad \sum_{XX} = \begin{bmatrix} \sum_{X(1)X(1)} & \cdots & \sum_{X(1)X(M)} \\ \vdots & \ddots & \vdots \\ \sum_{X(M)X(1)} & \cdots & \sum_{X(M)X(M)} \end{bmatrix},$$

$$\mathbf{h} = \left(\sum_{XX}\right)^{-1} \sum_{XY}$$

$$h_0 = \mu_Y - \mathbf{h}^T \mu_{\mathbf{X}} = \mu_Y - \left(\sum_{XY}\right)^T \left(\sum_{XX}\right)^{-1} \mu_{\mathbf{X}}$$

- MSE residual:

$$E\{(Y-\hat{Y})^2\} = E\{Y^2\} - E\{\hat{Y}^2\} = E\{(Y-\hat{Y})Y\}$$

$$E\{(Y-\hat{Y})^2\} = \sigma_Y^{\ 2} - \mathbf{h}^T \sum_{XY}$$