FIR Filter Synthesis by Windowing

- IIR Design from the continuous-time filters (mm2, mm3)
  - (*) Frequency-selective filters: lowpass, highpass ...
  - (*) Impulse Invariance Method

    $$h[n] = T_d h_c(nT_d); \quad \omega = \Omega T_d$$

  - (*) Bilinear Transformation Method

    $$s = \frac{2}{T_d} \frac{1 - z^{-1}}{1 + z^{-1}}; \quad \omega = 2\arctan\left(\frac{\Omega T_d}{2}\right)$$

  - (*) Butterworth, Chebyshev I.II, Elliptic filters ...

→ Specifications about magnitudes of desired filter ..

$$H(e^{j\omega}) = \underbrace{|H(e^{j\omega})|}_{\substack{\text{magnitude} \\ \text{(Amplitude, gain)}}} e^{\overbrace{j\measuredangle H(e^{j\omega})}^{\text{phase}}}$$

If a phase property, such as linear phase, is required besides the magnitude requirements, then

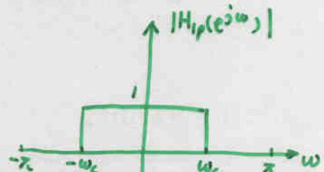
FIR Filter Design ....

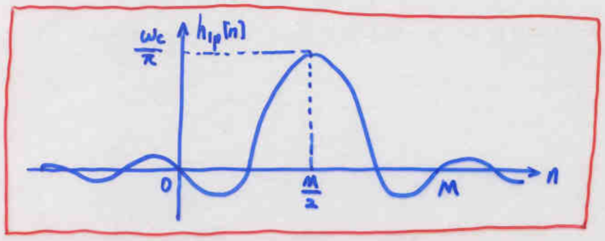
- Window Method
- Park-McClellan Method
- ....

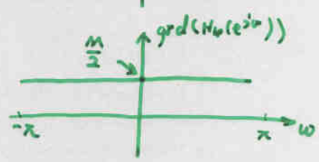
example 1 : Linear Phase lowpass filter

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\frac{M}{2}\omega} & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| \le \pi \end{cases}$$

$$h_{lp}[n] = \frac{\sin[\omega_c(n-\frac{M}{2})]}{\pi(n-\frac{M}{2})}$$







Linear Phase, but $\begin{cases} \text{not causal !} \\ \text{not FIR !} \end{cases}$

- Causal LTI system $\Rightarrow$ $h[n] = 0$ for $n < 0$
- FIR system $\Rightarrow$ $h[n] \neq 0$ for finite number of n

Design a sequence $w[n]$ — rectangular window

$$w[n] = \begin{cases} 1 & \text{for } 0 \le n \le M \\ 0 & \text{others} \end{cases}$$

then

$$\underbrace{w[n]\,h_{lp}[n]}_{} = \begin{cases} \frac{\sin[\omega_c(n-\frac{M}{2})]}{\pi(n-\frac{M}{2})} & 0 \le n \le M \\ 0 & \text{others} \end{cases}$$

M is integer, $h[n] \Rightarrow \begin{cases} \text{FIR} \\ \text{linear phase} \end{cases}$ (type I, type II forms).

How about the frequency response of $h[n]$?

$$H(e^{j\omega}) = \mathcal{F}\{w[n]\,h_{lp}[n]\}$$ (Modulation, Windowing Theorem) P.61

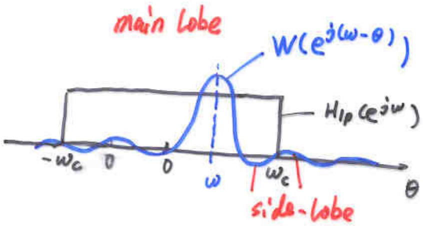
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} W(e^{j\theta})\, H_{lp}(e^{j(\omega-\theta)})\, d\theta$$

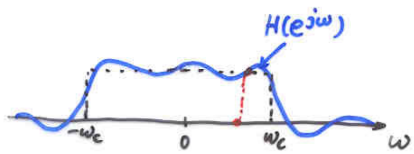
- If $w[n] = 1$ for all n, i.e., $W(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega+2k\pi)$

$$H(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\,\delta(\theta+0)\, H_{lp}(e^{j(\omega-\theta)})\, d\theta$$

$$= H_{lp}(e^{j(\omega-\theta)})\Big|_{\theta=0} = H_{lp}(e^{j\omega})$$

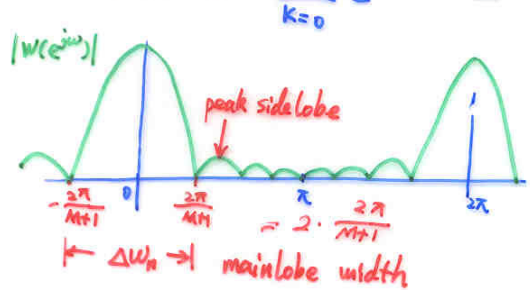
Doesn't truncate anything! But it seems that if $w[n]$ is chosen so that $W(e^{j\omega})$ is concentrated in a narrow band of frequency arround $\omega = 0$, then $H(e^{j\omega})$ will "look like" $H_{lp}(e^{j\omega})$





- If $w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{others} \end{cases}$ — window

$$W(e^{j\omega}) = \sum_{k=0}^{M} e^{-j\omega k} = e^{-j\frac{M}{2}\omega}\,\frac{\sin[\omega(\frac{M+1}{2})]}{\sin\frac{\omega}{2}}$$



$M\uparrow$, the amptitudes of the main lobe and side lobes $\uparrow$; the widths of them $\downarrow$ $\Rightarrow$ Gibbs phenomenon

- By tapering the window smoothly to zero at each end, the height of the side lobes can be diminished; But the cost is a wider main lobe and thus a wider transition at the discontinuity

  — Deal with Gibbs phenomenon

- Commonly Used Windows see Fig. 7.21 P469

  * Rectangular

$$w[n] = \begin{cases} 1 & 0 \le n \le M \\ 0 & \text{others} \end{cases}$$

  * Bartlett (triangular)

$$w[n] = \begin{cases} \frac{2n}{M} & 0 \le n \le \frac{M}{2} \\ 2 - \frac{2n}{M} & \frac{M}{2} < n \le M \\ 0 & \text{others} \end{cases}$$

  * Hanning

$$w[n] = \begin{cases} 0.5 - 0.5\cos\frac{2\pi n}{M} & 0 \le n \le M \\ 0 & \text{others} \end{cases}$$

  * Hamming

$$w[n] = \begin{cases} 0.54 - 0.46\cos\frac{2\pi n}{M} & 0 \le n \le M \\ 0 & \text{others} \end{cases}$$

  * Blackman

$$w[n] = \begin{cases} 0.42 - 0.5\cos\frac{2\pi n}{M} + 0.08\cos\frac{4\pi n}{M} & 0 \le n \le M \\ 0 & \text{others} \end{cases}$$

• Property of Windows

$$w[n] = \begin{cases} w[M-n] & 0 \le n \le M \\ 0 & \text{others} \end{cases}$$ symmetric about $\frac{M}{2}$

$\Downarrow$ see P297

$$W(e^{jw}) = \underbrace{W_e(e^{jw})}_{\text{even, real function of } \omega} e^{-j\frac{M}{2}\omega}$$

$$H(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W(e^{j(\omega-\theta)})\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\theta})\, W_e(e^{j(\omega-\theta)})\, e^{-j\frac{M}{2}(\omega-\theta)}\, d\theta$$

$$= A_e(e^{jw})\, e^{-j\frac{M}{2}\omega}$$

where $A_e(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_e(e^{j\theta})\, W_e(e^{j(\omega-\theta)})\, d\theta$

$$H_d(e^{jw}) = H_e(e^{jw})\, e^{-j\frac{M}{2}\omega}$$

$IIR \xrightarrow{\text{windows}} FIR$

• If $h_d[M-n] = h_d[n]$ (symmetric at $\frac{M}{2}$),

$h[n] = h_d[n]\, w[n]$ is also symmetric at $\frac{M}{2}$

i.e., $h[n]$ is a Type I or Type II linear phase FIR system.

• If $h_d[M-n] = -h_d[n]$ (antisymmetric at $\frac{M}{2}$)

$h[n] = h_d[n]\, w[n]$ is also antisymmetric at $\frac{M}{2}$

i.e., $h[n]$ is a Type III or Type IV linear phase FIR system.

$$H(e^{jw}) = jA_o(e^{jw})\, e^{-j\frac{M}{2}\omega}$$

- Kaiser Window Method

(*) A near-optimal window that makes the trade-off between the main-lobe width and side-lobe area (Kaiser 1966)

$$\boxed{w[n] = \begin{cases} \dfrac{I_0\left[\beta\left(1-\left[(n-\alpha)/\alpha\right]^2\right)^{\frac{1}{2}}\right]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{others} \end{cases}}$$

Two Parameters $\begin{cases} (\alpha = \frac{M}{2}) \text{ the length } (M+1) \\ \text{shape parameter } \beta \end{cases}$

An empirical method to determine M and β

* precondition $\begin{cases} \text{passband cutoff frequency } \omega_p \\ \text{stopband } \text{------} \ \omega_s \\ \text{peak approximation err } \delta \end{cases}$

$$* \quad \boxed{\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07886(A-21) & 21 \le A \le 50 \\ 0.0 & A < 21 \end{cases} \quad \text{where } A = -20\log_{10}\delta}$$

$$* \quad \boxed{M = \frac{A-8}{2.285\,\Delta\omega} \qquad \text{where } \Delta\omega = \omega_s - \omega_p}$$

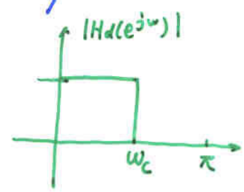
$\boxed{\text{Example 1:}}$ FIR Lowpass filter Design

Design a FIR lowpass filter, such that

$$0.89125 \le |H(e^{j\omega})| \le 1 \qquad 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.17783 \qquad 0.3\pi \le \omega \le \pi$$

Step 1: Obtain a ideal lowpass filter $H_d(e^{j\omega})$



The cutoff frequency $\omega_c = \dfrac{\omega_p + \omega_s}{2}$

$$= \frac{0.2\pi + 0.3\pi}{2} = 0.25\pi$$

Then

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\frac{M}{2}\omega} & |\omega_a| \le 0.25\pi \\ 0 & \text{others} \end{cases}$$

$$h_d[n] = \mathcal{F}^{-1}\{H_d(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{\sin \omega_c (n - \frac{M}{2})}{\pi (n - \frac{M}{2})} \qquad -\infty \le n \le \infty$$

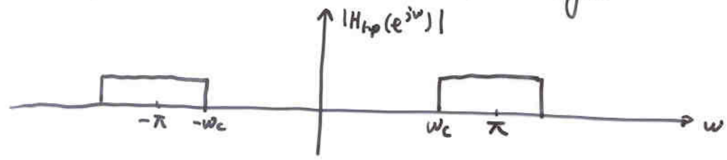
Step 2: Truncate the ideal lowpass filter by windows.

$$h[n] = h_d[n]\, w[n] = \begin{cases} \dfrac{\sin \omega_c (n-\alpha)}{\pi (n-\alpha)} \cdot \dfrac{I_0[\beta(1-[(n-\alpha)/\alpha]^2)^{\frac{1}{2}}]}{I_0(\beta)} & 0 \le n \le M \\ 0 & \text{others} \end{cases}$$

where $A = -20\log_{10}(0.1075)$ $\quad \Delta\omega = \omega_s - \omega_p = 0.1\pi$

$$\beta = \cdots 5.653 \qquad M = 37.$$

Example 2: FIR HIGHPASS FILTER Design



Step 1: Obtain the ideal highpass filter

$$H_{hp}(e^{jw}) = \begin{cases} 0 & 0 \le |w| \le w_c \\ e^{-jw\frac{M}{2}} & w_c \le |w| \le \pi \end{cases}$$

$$H_{hp}(e^{jw}) = e^{-jw\frac{M}{2}} - H_{lp}(e^{jw})$$

$$h_{hp}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{hp}(e^{jw})\, e^{jwn}\, dw$$

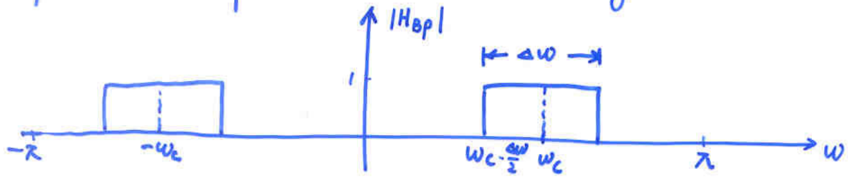
$$= \underbrace{\frac{\sin \pi(n-\frac{M}{2})}{\pi(n-\frac{M}{2})}}_{= \begin{cases} 1 & n = \frac{M}{2} \\ 0 & \text{others} \end{cases}} - \underbrace{\frac{\sin w_c(n-\frac{M}{2})}{\pi(n-\frac{M}{2})}}_{\text{ideal lowpass filter}} \qquad -\infty < n < \infty$$

$h_{hp}[n]$ is symmetric at $\frac{M}{2}$ point.

Step 2: Truncate the ideal highpass filter by windows

$$h[n] = h_{hp}[n]\, w[n] = \cdots\cdots$$

Example 3 : Bandpass FIR Filter Design



step 1 : obtain the ideal bandpass filter

$$H_{BP}(e^{j\omega}) = \begin{cases} e^{-j\frac{M}{2}\omega} & \begin{cases} -\omega_c - \frac{\Delta\omega}{2} < \omega < -\omega_c + \frac{\Delta\omega}{2} \\ \omega_c - \frac{\Delta\omega}{2} < \omega < \omega_c + \frac{\Delta\omega}{2} \end{cases} \\ 0 & \text{others} \end{cases}$$

$$h_{BP}[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{BP}(e^{j\omega}) e^{j\omega n}\, d\omega$$

$$= \frac{\sin\left((\omega_c + \frac{\Delta\omega}{2})(n - \frac{M}{2})\right) - \sin\left((\omega_c - \frac{\Delta\omega}{2})(n - \frac{M}{2})\right)}{\pi(n - \frac{M}{2})}$$

Assume $\omega_c = \frac{\pi}{2}$, $\Delta\omega = \frac{\pi}{2}$, then

$$h_{BP}[n] = \frac{\sin\left(\frac{3\pi}{4}(n - \frac{M}{2})\right) - \sin\left(\frac{\pi}{4}(n - \frac{M}{2})\right)}{\pi(n - \frac{M}{2})}$$

Step 2 : Truncate the $h_{BP}[n]$ by windows

- Rectangular window ($M = \cdots$)
- Hanning window ($M = \ldots$)
- Kaiser window (M, β)