Modeling of Electromechanical Systems

Werner Haas, Kurt Schlacher and Reinhard Gahleitner
Johannes Kepler University Linz,
Department of Automatic Control,
Altenbergerstr.69, A–4040 Linz,
email: werner@regpro.mechatronik.uni-linz.ac.at

Preliminary Version, January 17, 2000

Contents

1 Motivation

Why should we study modeling?

For the answer we need some basic concepts and definitions. We start with a definition for the term system: A system is characterized by the fact that we can say what belongs to it and what does not. We can specify how it interacts with its environment Hence it can be observed and controlled.

- There are variables that are generated by the environment and that influence the behavior of the system. These are called the inputs of the system.
- There are other variables that are determined by the system and that in turn influence the behavior of its environment. These are called the outputs of the system.

This leads to a possible definition for the term system [1]:

A system is a potential source of data

This leads to a definition for the term experiment:

An experiment is the process of extracting data from a system by exerting it through its inputs

To perform an experiment on the system means to apply a set of external conditions to the inputs and to observe the reaction of the system to these inputs by recording the behavior of the outputs. The major disadvantage with real systems is the fact that these systems are under the influence of inaccessible inputs (so called disturbances) and a number of useful outputs are not accessible through measurements. The definition of a system and an experiment gives a way to define the term model:

A model M for a system S and an experiment E is anything to which E can be applied in order to answer questions about S

In the present course, we concentrate ourselves to a subclass of models which are called mathematical models. This is a description of the relationship among the system variables in terms of mathematical equations. By performing experiments, we collect knowledge about a system.

Modeling means the process of organizing knowledge about a given system

In the beginning, this knowledge is unstructured. By understanding what are causes and what are effects, we organize the knowledge. In fact, we are engaged in a process of modelling. The major task in which a model is to be used has basic implications on the choice of the particular form of a model. In other words, a model can be considered as a specialized tool, developed for a particular application. Of course, such approach leads to different models for different uses of the same system. In particular a control engineer uses a model for the development of control algorithms. Thus models, for control reasons, should be as simple as possible. Often a model which takes into account the first order effects is adequate. A well designed controller suppresses the second order effects. Another point is that the effort of the controller development is often related to the complexity of the model. Simulation engineers want to develop models which fit the experimental data in an appropriate way. This approach may lead to more complex models.

There are different types of mathematical models [1], [6]. We concentrate our investigations to lumped parameter models which are described by ordinary differential equations of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t).$$

Moreover, we are interested especially in electromechanical systems.

How do we perform the process of modeling?

1. We start with a physical description of the dynamical systems. This includes a discussion of physical effects which should be taken into account.
2. Calculation of a single quantity which determines the time behavior of the dynamical system. Later, we will show that this quantity is called the extended Lagrangian.
3. Derivation of the equations of motion. A computer algebra program (MAPLE V) is used to derive the mathematical model in an automatic way.
4. Simulation code (MATLAB or DYNAST) for the model is generated automatically.
5. Simulation

Summary

The derivation of the equations of motion (the mathematical model) can be obtained from variational principles applied to energy functions. There exists a well established common terminology for all type of systems, whether electrical, mechanical, magnetic, etc., by defining energy functions in terms of the generalized coordinates. Then by the use of a single fundamental postulate, e.g. Hamilton's principle, the equations of motion are determined. The variational approach is quite formal analytically and as a result insight into physical processes can be lost in the mathematical procedure. Nevertheless, if the method is properly understood, physical insight can be gained due to the generality of the method. There are a number of different energy functions (e.g. the Lagrangian, the total energy) which can be used as a energy function.

In this course the modelling of purely mechanical systems is mainly based on the Lagrangian which is a function of the generalized coordinates and the associated velocities. If all forces are derivable from a potential, then the time behavior of the dynamical systems is completely determined. For simple mechanical systems, the Lagrangian is defined as the difference of the kinetic energy and the potential energy.

There exists a similar approach for electrical system. By means of the electrical coenergy and well defined power quantities, the equations of motions are uniquely defined. The currents of the inductors and the voltage drops across the capacitors play the role of the generalized coordinates. All constraints, for instance caused by the Kirchhoff laws, are eliminated from the considerations.

In consequence, we have quantities (kinetic and potential energy, generalized forces) which determine the mechanical part and quantities (coenergy, powers) for the description of the electrical part. This offers a combination of the mechanical and electrical parts by means of an energy approach. As a result, we get an extended Lagrangian.

2 Mechanical Systems

2.1 Derivation of the Lagrange Equations

In the case of systems of N particles we need, in general, $3N$ coordinates to specify the position of all particles. If there exist constraints, then the number of coordinates actually needed to describe the

system is reduced. For instance, for the specification of a rigid body, we need six coordinates, three for the reference point and three for the orientation. In general, a certain minimum number n of coordinates, called the degrees of freedom, is required to specify the configuration. Usually, these coordinates are denoted by q_i and are called generalized coordinates. The coordinate vector

$$\mathbf{x}_i, \qquad i = 1, \ldots, N \tag{1}$$

of a specific particle and the generalized coordinates are related by equations of the form

$$\mathbf{x}_i = \mathbf{x}_i \left(q_1, q_2, \ldots, q_n, t \right) . \tag{2}$$

The time t appears explicitly in the case of moving constraints, such as a particle is constrained to move on a surface which itself is moving in a predefined way. The choice of the generalized coordinates is usually somewhat arbitrary, but in general each individual energy storage element of the system have a set of generalized coordinates. For a dynamic system the generalized coordinates do not completely specify the system and an additional set of dynamic variables equal in number to the generalized coordinates must be used. These dynamic variables can be the first time derivatives of the generalized coordinates, the velocities, or can be a second set variables (e.g. the generalized momenta).

In order to find the differential equations of motion in terms of the generalized coordinates, we use the energy of the system. The kinetic energy T in terms of Cartesian coordinates is given by

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{v}_i^T \mathbf{v}_i, \qquad \frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i. \tag{3}$$

Remark 1 *It is assumed that masses are not functions of the velocities or coordinates.*

From the relation (2), we obtain

$$\mathbf{v}_i = \sum_{j=1}^{n} \frac{dq_j}{dt} \frac{\partial \mathbf{x}_i}{\partial q_j} + \frac{\partial \mathbf{x}_i}{\partial t}, \tag{4}$$

and

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{x}_i}{\partial q_j}, \tag{5}$$

which gives

$$\frac{d}{dt} \left(\mathbf{v}_i^T \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) = \frac{d}{dt} \left(\mathbf{v}_i^T \frac{\partial \mathbf{x}_i}{\partial q_j} \right) = \left(\frac{d}{dt} \mathbf{v}_i \right)^T \frac{\partial \mathbf{x}_i}{\partial q_j} + \mathbf{v}_i^T \frac{\partial \mathbf{v}_i}{\partial q_j} \tag{6}$$

after multiplication with $\mathbf{v}_i^T$ and differentiation with respect to t. This leads to

$$\frac{1}{2}\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}\left(\mathbf{v}_i^T\mathbf{v}_i\right) = \left(\frac{d}{dt}\mathbf{v}_i\right)^T\frac{\partial \mathbf{x}_i}{\partial q_j} + \frac{1}{2}\frac{\partial}{\partial q_j}\left(\mathbf{v}_i^T\mathbf{v}_i\right). \tag{7}$$

Next, we multiply by m_i and make use of the relation

$$\mathbf{F}_i = m_i\frac{d}{dt}\mathbf{v}_i. \tag{8}$$

Hence, by summing over all i, we find

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = \sum_{i=1}^{N}\mathbf{F}_i^T\frac{\partial \mathbf{x}_i}{\partial q_j} + \frac{\partial T}{\partial q_j}. \tag{9}$$

Here the kinetic energy $T(\dot{q})$ is assumed to be a function of $\dot{q}$. The expression

$$\sum_{i=1}^{N}\mathbf{F}_i^T\frac{\partial \mathbf{x}_i}{\partial q_j} = Q_j, \qquad j = 1,\dots,n \tag{10}$$

defines the generalized forces Q_j. Hence, we obtain the result

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} = Q_j + \frac{\partial T}{\partial q_j}, \qquad j = 1,\dots,n. \tag{11}$$

These are differential equations of motion in the generalized coordinates q_j. They are known as Lagrange equations of motion. If part of the generalized forces are conservative, then some Q_j can be expressed as

$$Q_j = \sum_{i=1}^{N}\mathbf{F}_i^T\frac{\partial \mathbf{x}_i}{\partial q_j} = \sum_{i=1}^{N}\left(-\left(\frac{\partial V}{\partial \mathbf{x}_i}\right)^T\frac{\partial \mathbf{x}_i}{\partial q_j}\right) + Q_j^e \tag{12}$$

and finally

$$Q_j = -\frac{\partial V}{\partial q_j} + Q_j^e. \tag{13}$$

V is called the potential energy function and the Q_j^e are generalized forces not derivable from a potential energy function V. Now, the equations can be written more compactly by defining the Lagrangian

$$L = T - V \tag{14}$$

which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^e, \qquad j = 1, \dots, n. \tag{15}$$

The Lagrange equations have been derived from Newton's laws. In fact, they are a redefinition of Newton's laws written out in terms of appropriate variables such that constraint forces are eliminated from considerations. The dynamical system is defined by a single function L, at least if all forces are conservative. The general procedure for finding the differential equations of motion for a system is as follows:

1. Select a suitable set of coordinates to represent the configuration of the system.
2. Obtain the kinetic energy T as a function of these coordinates and their time derivatives.
3. If the system is conservative, find the potential energy V as a function of the coordinates, or, if the system is not conservative, find the generalized forces Q_j^e.
4. The differential equations of motion are then given by equations (15).

Remark 2 *The application of the Lagrangian formulation is not restricted to mechanical systems. So, there exists Lagrangians which are not defined as the difference between the kinetic and potential energy.*

Remark 3 *The Lagrangian function determines the equations of motion uniquely, the converse of this fact is not true.*

Remark 4 *The Lagrange equations were derived without specifying a particular generalized coordinate system. Hence, they are also valid in other coordinate systems. Lagrange's equations are coordinate independent.*

Remark 5 *The Lagrangian function is a so called state function. Its value at a given instant of time is given by the state of the system at that time, and not on the history.*

Remark 6 *The Lagrangian depends on the generalized coordinates q, the associated velocities $\dot{q}$, and the time t.*

As mentioned above, external forces can be subdivided into two groups:

- The first group consists of forces **F** which are given by a potential function

$$\mathbf{F} = -\frac{\partial V}{\partial \mathbf{x}_i}.$$

- The second group is formed by non potential forces.

Suppose a non potential force which is a function of the velocity and that the force is directed opposite to the velocity of the particle, e.g.

$$F_D = -gv, \tag{17}$$

with $g > 0$. Hence, the force does negative work and this leads to energy loss. Such forces are called dissipative. From the relations

$$Q_j = \sum_{i=1}^{N} \mathbf{F}_i^T \frac{\partial \mathbf{x}_i}{\partial q_j} \quad \Rightarrow \quad Q_D = F_D \frac{\partial x}{\partial q}, \qquad \text{and} \qquad \frac{\partial x}{\partial q} = \frac{\partial v}{\partial \dot{q}} \tag{18}$$

we get

$$Q_D = F_D \frac{\partial x}{\partial q} = -gv \frac{\partial v}{\partial \dot{q}}. \tag{19}$$

Let us define the so called dissipative function or Rayleigh potential P^{R} with

$$Q_D = -\frac{\partial}{\partial \dot{q}} P^{\mathrm{R}}. \tag{20}$$

The combination of the relations ([19](#)) and ([20](#)) leads to

$$\frac{\partial}{\partial \dot{q}} P^{\mathrm{R}} = gv \frac{\partial v}{\partial \dot{q}}, \qquad P^{\mathrm{R}} = \int_0^v gv' \, \mathrm{d}v' = g\frac{v^2}{2}. \tag{21}$$

The prime denotes the variable of integration. If $g > 0$ is positive, then P^{R} is a positive function. The modified Lagrange equations now read

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} + \frac{\partial P^{\mathrm{R}}}{\partial \dot{q}_j} = 0, \qquad j = 1, \dots, n. \tag{22}$$

Of course there exist other dissipative forces not related to an equation like ([17](#)).

2.1.1 Example I

Consider the mass–spring system given in figure ([1](#)).
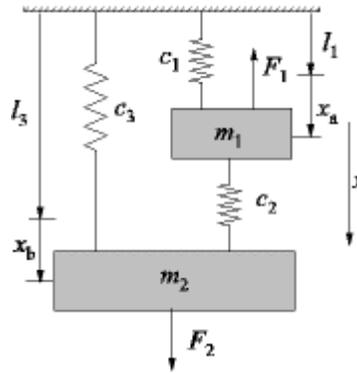
Figure 1: Mechanical example.

In the equilibrium (zero forces F_1 and F_2, the system is forced by the Earth gravitational force *mg*) the length of the springs are given with l_1, l_2, and l_3. Then, the coordinates x_a and x_b measure the deviation from the equilibrium. If x_a and x_b are specified, then the geometric configuration of the system is completely determined. So, we have found a set of generalized coordinates x_a and x_b and their associated velocities v_a and v_b. Referring to equation ([14](#)) and equation ([15](#)), we start with the calculation of the kinetic energy and find

$$T = \frac{1}{2}m_1 v_a^2 + \frac{1}{2}m_2 v_b^2.$$

Next, the potential energy is given as

$$V = \frac{1}{2}c_1 x_1^2 + \frac{1}{2}c_2 x_2^2 + \frac{1}{2}c_3 x_3^2 - m_1 g x_1 - m_2 g x_2$$

with the lengths x_i of the springs. With the geometric relations

$$x_1 = x_a + l_1, \qquad x_2 = x_b + l_3 - x_a - l_1 = x_b - x_a + l_2, \qquad x_3 = x_b + l_3$$

the Lagrangian follows as

$$L = T - V =$$

$$\frac{1}{2}m_1 v_a^2 + \frac{1}{2}m_2 v_b^2 -$$

$$-\frac{1}{2}c_1 \left(x_a + l_1\right)^2 - \frac{1}{2}c_2 \left(x_b - x_a + l_2\right)^2 - \frac{1}{2}c_3 \left(x_b + l_3\right)^2 + m_1 g \left(x_a + l_1\right) + m_2 g \left(x_b + l_3 - x_a - l_1\right)$$

The generalized forces are

$$Q_1^e = -F_1, \qquad Q_2^e = F_2.$$

The application of the Lagrange Formalism leads to the equations of motion.

$$x_a : m_1 \frac{dv_a}{dt} + c_1 (x_a + l_1) - c_2 (x_b - x_a + l_2) - m_1 g + m_2 g = -F_1$$

$$x_b : m_2 \frac{dv_b}{dt} + c_2 (x_b - x_a + l_2) + c_3 (x_b + l_3) - m_2 g = F_2.$$

2.1.2 Example II

Consider the motor–shaft–load system given in figure (2).
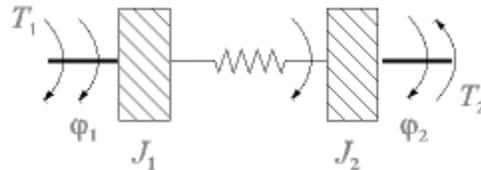
Figure 2: Torsion drive.

The motor is represented by the rotating inertia J_1 and the torque generated by the motor is a given function T_1. The load represented by the inertia J_2 is coupled to the motor by means of an elastic shaft with stiffness c. In addition, there is a load torque T_2. The coordinates φ_1 and φ_2 determines the geometric configuration of the system completely. Therefore the generalized coordinates are φ_1 and φ_2 and their associated angular velocities ω_1 and ω_2. Referring to equation (14) and equation (15), we start with the calculation of the kinetic energy and find

$$T = \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} J_2 \omega_2^2.$$

Next, the potential energy is given as

$$V = \frac{1}{2} c (\varphi_1 - \varphi_2)^2$$

with the angular φ_1 - φ_2 of the torsion spring. The Lagrangian follows as

$$L = T - V = \frac{1}{2} J_1 \omega_1^2 + \frac{1}{2} J_2 \omega_2^2 - \frac{1}{2} c (\varphi_1 - \varphi_2)^2 .$$

The generalized forces are

$$Q_1^e = T_1, \qquad Q_2^e = -T_2.$$

The application of the Lagrange Formalism leads to the equations of motion.

$$\varphi_1 : J_1 \frac{d\omega_1}{dt} + c_1 (\varphi_1 - \varphi_2) = T_1$$

$$\varphi_2 : J_1 \frac{d\omega_2}{dt} - c_1 (\varphi_1 - \varphi_2) = -T_2.$$

2.2 Variational Principle and Lagrange's Equations

There exists an alternative way of deriving Lagrange's equations which gives new insights. This method is based on Hamilton's variational principle: The motion of a system takes place in such a way that the integral

$$\int_{t_1}^{t_2} (T + W)\, dt \tag{23}$$

is an extremum. The work W of the external forces is given by

$$W = \sum_{i=1}^{N} \mathbf{F}_i^T \mathbf{x}_i \ . \tag{24}$$

In other words, Hamilton's principle says that out of all possible ways a system can change within a given finite time t_2 - t_1, that particular motion which will occur, for which the integral is either a maximum or a minimum. The statement can be expressed in mathematical terms as

$$\delta \int_{t_1}^{t_2} (T + W)\, dt = 0 \tag{25}$$

in which δ denotes a small variation. This variation results from taking different paths of integration by varying the generalized coordinates q_j. Note, no variation takes place with respect to the time t. Caused by the variations δq_j we have virtual displacements $\delta \mathbf{x}_i$ of the coordinates $\mathbf{x}_i$. This leads to

$$\delta W = \sum_{i=1}^{N} \mathbf{F}_i^T \delta \mathbf{x}_i = \sum_{i=1}^{N} \mathbf{F}_i^T \sum_{j=1}^{n} \frac{\partial \mathbf{x}_i}{\partial q_j} \delta q_j \tag{26}$$

and

$$\delta W = \sum_{j=1}^{n} \sum_{i=1}^{N} \mathbf{F}_i^T \frac{\partial \mathbf{x}_i}{\partial q_j} \delta q_j = \sum_{j=1}^{n} Q_j \delta q_j. \tag{27}$$

A first fact is that the product

$$Q_j \delta q_j \tag{28}$$

is the work done on the system by the external forces, when the coordinates q_j change a virtual amount δq_j. The other generalized coordinates are remaining constant. For example, if the system is a rigid body, the work done by the external forces when the body turns through an angle $\delta\varphi$ about a given axis is

$$M \delta\varphi, \tag{29}$$

where M is the torque about the axis. In this case, M is the generalized force associated with the angle φ. The combination of (25) and (27) gives

$$\delta \int_{t_1}^{t_2} T \mathrm{d}t + \int_{t_1}^{t_2} \sum_{j=1}^{n} Q_j \delta q_j \mathrm{d}t = 0. \tag{30}$$

Let Q_j be a generalized force which is derivable from a potential energy function V. In this case, we get by integration by parts

$$\int_{t_1}^{t_2} \sum_{j=1}^{n} Q_j \delta q_j \mathrm{d}t = - \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j \mathrm{d}t = - \int_{t_1}^{t_2} \delta V \, \mathrm{d}t \ . \tag{31}$$

The combination of (30) and (31) gives

$$\delta \int_{t_1}^{t_2} (T - V) \, \mathrm{d}t + \int_{t_1}^{t_2} \sum_{j} Q_j^{\mathrm{e}} \delta q_j \mathrm{d}t = 0 \ , \tag{32}$$

where the summation goes over the generalized forces which are not derivable from a potential function. The Lagrangian

$$L = T - V \tag{33}$$

is a function of q_j and $\dot{q}$. We have

$$\delta \int_{t_1}^{t_2} L \mathrm{d}t = \int_{t_1}^{t_2} \delta L \mathrm{d}t = \int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) \mathrm{d}t \tag{34}$$

and by integration by parts

$$\delta \int_{t_1}^{t_2} L \mathrm{d}t = \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \delta q_j \mathrm{d}t + \left[\sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \delta q_j \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_{j=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_j} \delta q_j \mathrm{d}t. \tag{35}$$

For fixed values of the limits t_1 and t_2, the variation $\delta q_j = 0$ at time t_1 and t_2. Hence, we get

$$\int_{t_1}^{t_2} \sum_{j=1}^{n} \left(\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_j} + Q_j^{\mathrm{e}} \right) \delta q_j \mathrm{d}t = 0. \tag{36}$$

If all the generalized coordinates are independent, then their variations are all independent, too. Therefore, each term in the bracket must vanish in order that the integral itself vanishes. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^{\mathrm{e}}, \qquad j = 1, \dots, n. \tag{37}$$

Remark 7 *Similar equations were first derived by Euler for the general mathematical variational problem. Therefore, the equations (37) are called Euler Lagrange equations, too.*

Remark 8 *The variational approach leads to the Euler–Lagrange equations even when relation (23) does not give a minimum. The minimum requirement is always satisfied if L = T - V holds and V is independent of the velocities (or if V depends linearly on the velocity).*

2.3 State Functions

As mentioned above the value of the Lagrangian at a given instant of time is a function given by the state of the system at that time, and not on the history. Such functions are called state functions. Examples are the total energy of the system and other closely related functions. The are of central importance in the characterization of physical systems. For example, let dW be a differential change in energy produced by a differential displacement dq in the variable q. Then we have

$$\mathrm{d}W = Q\mathrm{d}q \tag{38}$$

with the generalized force Q – see equation (28) also. The product of the variables Q and q describes an energy relation, which is usually a state function. It contains much valuable information about the system. Unfortunately some physical effects (dissipation, hysteresis, inputs) must be excluded from systems if they are to be described by state function. So, we restrict our attention to conservative systems. This is not a serious drawback, because this formulation is mainly used for the coupling of electrical and mechanical part. Fortunately, these couplings are derivable from state functions.

The state of the dynamical system can be described either by n generalized coordinates q_i and its time derivatives $\dot{q}_i$ or by the q_i and the n generalized momenta p_i. The associated 2n dimensional space is called the phase space. A pair q_i and p_i is called canonically conjugate variables. Associated with every set of independent variable q_i and p_i is a set of dependent variables Q_i and $\dot{q}_i$. So, for a mechanical system we have four different kind of variables:

- q, the generalized mechanical coordinate, it is also called mechanical displacement,
- $\dot{q}$, the generalized mechanical velocity, it is also called mechanical velocity,
- Q, the generalized mechanical force, a mechanical force depends upon the position only

$$Q = K\left(q\right)q.$$

- p, the generalized mechanical momenta – see equation (47), a mechanical momenta depends usually upon the velocity only

$$p = M\left(\dot{q}\right)\dot{q}.$$

We have mentioned that there are Lagrangian which cannot be expressed as the difference of the kinetic

and potential energy. Nevertheless it is possible to decompose L as the difference of two functions. To show this we express the differential of the Lagrangian

$$L\left(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t\right) \tag{41}$$

as

$$\mathrm{d}L = \sum_{j=1}^{n} \frac{\partial L}{\partial q_j} \mathrm{d}q_j + \sum_{j=1}^{n} \frac{\partial L}{\partial \dot{q}_j} \mathrm{d}\dot{q}_j + \frac{\partial L}{\partial t} \mathrm{d}t. \tag{42}$$

From this expression L can be calculated by integration. Since L is a state function, a arbitrary path for the integration can be chosen. For example the $\dot{q}_j$ are constant for integration with respect to q_j and the q_j are constant for integration with respect to $\dot{q}_j$. Furthermore, these integrations can be performed for a specific value of t. We get

$$L =$$

$$\int_{0,\ldots,0}^{q_1,\ldots,q_n} \sum_{j=1}^{n} \frac{\partial L\left(q'_1, \ldots, q'_n, 0, \ldots, 0, t\right)}{\partial q_j} \mathrm{d}q'_j +$$

$$\int_{0,\ldots,0}^{\dot{q}_1,\ldots,\dot{q}_n} \sum_{j=1}^{n} \frac{\partial L\left(q_1, \ldots, q_n, \dot{q}'_1, \ldots, \dot{q}'_n, t\right)}{\partial \dot{q}_j} \mathrm{d}\dot{q}'_j \tag{43}$$

and L is decomposed in two functions. The first function is exactly the definition for the negative of the potential energy. Therefore a generalized force Q_j associated to a potential is defined as

$$Q_j = \frac{\partial L\left(q'_1, \ldots, q'_n, 0, \ldots, 0, t\right)}{\partial q_j} \tag{44}$$

and the potential energy is defined as

$$V = - \int_{0,\ldots,0}^{q_1,\ldots,q_n} \sum_{j=1}^{n} Q_j\left(q'_1, \ldots, q'_n, t\right) \mathrm{d}q'_j. \tag{45}$$

This clarifies the introduction of the potential energy in equation ([31](#)). The second term

$$\int_{0,\ldots,0}^{\dot{q}_1,\ldots,\dot{q}_n} \sum_{j=1}^{n} \frac{\partial L\left(q_1, \ldots, q_n, \dot{q}'_1, \ldots, \dot{q}'_n, t\right)}{\partial \dot{q}_j} \mathrm{d}\dot{q}'_j \tag{46}$$

is a function of the final values of q_j and the velocities. This acts as a definition of the generalized momenta

$$p_j = \frac{\partial L\left(q_1,\ldots,q_n,\dot{q}_1',\ldots,\dot{q}_n',t\right)}{\partial \dot{q}_j}$$

(47)

and the so called kinetic coenergy

$$T' = \int_{0,\ldots,0}^{\dot{q}_1,\ldots,\dot{q}_n} \sum_{j=1}^{n} p_j\left(\dot{q}_1',\ldots,\dot{q}_n',t\right) \, d\dot{q}_j'.$$

(48)

Remark 9 *At this point the reason for this terminology seems to be artificial. The analogous discussion for electrical systems shows that the introduction of the kinetic coenergy is a direct consequence of the definition of the magnetic coenergy.*

Remark 10 *In this concept the definition of the kinetic energy has the general form*

$$T = \int \dot{q}\,dp$$

(49)

whereas the definition of the kinetic energy has the general form

$$T' = \int p\,d\dot{q}.$$

(50)

As a consequence, the Lagrangian becomes simply

$$L = T' - V.$$

(51)

If the masses of a mechanical system are constant, then the kinetic coenergy T' and the kinetic energy T are equal. For example suppose a mass m with velocity $\dot{q}$. The momenta is given as $p = m\dot{q}$ and we obtain the kinetic coenergy

$$T' = \int_0^{\dot{q}} p\left(\dot{q}'\right) d\dot{q}' = \frac{m\dot{q}^2}{2},$$

(52)

which is equal to the kinetic energy T.

2.3.1 Example I

The example should illustrate the calculation of the kinetic and potential energies. Consider the mass–spring system given in figure (3).
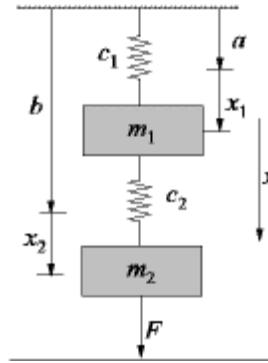
Figure 3: Example.

In the equilibrium (zero force F) the length of the springs are given with a and b. Then, the coordinates x_1 and x_2 measure the deviation from the equilibrium. If x_1 and x_2 are specified, then the geometric configuration of the system is completely determined. So, we have found the generalized coordinates x_1 and x_2 and their associated velocities v_1 and v_2. To find the potential energy the equation (45) is used, thus

$$V = - \int\limits_{0,0}^{x_1, x_2} \sum_{j=1}^{2} Q_i \left(x_1', x_2', t \right) dx_j'$$

where

$$Q_1 = -c_1 x_1' + c_2 \left(x_2' - x_1' \right), \qquad Q_2 = -c_2 \left(x_2' - x_1' \right).$$

The potential energy is

$$V = - \int\limits_{0,0}^{x_1, x_2} \left(-c_1 x_1' + c_2 \left(x_2' - x_1' \right) \right) dx_1' - c_2 \left(x_2' - x_1' \right) dx_2'.$$

This integral is evaluated by holding $x_2' = 0$ and displacing x_1 from 0 to x_1, then holding $x_1' = x_1$ and displacing x_2 from 0 to x_2. This results in

$$V = (c_1 + c_2) \int\limits_0^{x_1} x_1' dx_1' + \int\limits_0^{x_2} c_2 \left(x_2' - x_1 \right) dx_2'$$

$$V = \frac{(c_1 + c_2)}{2} x_1^2 + \frac{c_2}{2} x_2^2 - c_2 x_2 x_1$$

$$V = \frac{c_1}{2} x_1^2 + \frac{1}{2} c_2 \left(x_2 - x_1 \right)^2.$$

The kinetic coenergy can now be derived using equation (48), which is

$$T' = \int\limits_{0,0}^{\dot{x}_1,\dot{x}_2} \sum_{j=1}^{2} p_j \left(\dot{x}'_1, \dot{x}'_2 \right) d\dot{x}'_j.$$

The momenta are

$$p_1 = m_1 \dot{x}_1, \qquad p_2 = m_2 \dot{x}_2$$

and we get

$$T' = \int\limits_{0,0}^{\dot{x}_1,\dot{x}_2} m_1 \dot{x}'_1 d\dot{x}'_1 + \int\limits_{0,0}^{\dot{x}_1,\dot{x}_2} m_2 \dot{x}'_2 d\dot{x}'_2.$$

The line integral is

$$T' = \int\limits_{0,0}^{\dot{x}_1,0} m_1 \dot{x}'_1 d\dot{x}'_1 + \int\limits_{\dot{x}_1,0}^{\dot{x}_1,\dot{x}_2} m_2 \dot{x}'_2 d\dot{x}'_2 = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2.$$

2.3.2 Example II

The upper point of the ideal pendulum of length l is constrained to move at a constant angular velocity ω around a circle of radius r.
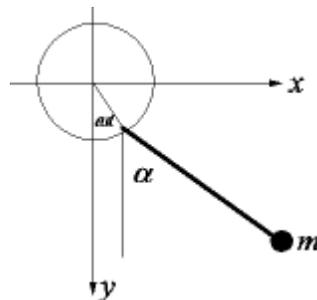
Figure 4: Pendulum on a circle.

At time $t = 0$ the upper point of the pendulum is located at the bottom of its circular path. We assume that there is no friction. If α is specified, then the position of the pendulum is completely determined. So, α is the generalized coordinate and $\dot{\alpha}$ the associated velocity. In terms of the Cartesian–coordinate system the kinetic coenergy of the mass m is given by

$$T' = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2.$$

The coordinates x and y are expressed in terms of the generalized coordinate, thus

$$x = r \sin (\omega t) + l \sin \alpha, \qquad y = r \cos (\omega t) + l \cos \alpha.$$

Using the first time derivatives

$$\dot{x} = r\omega \cos(\omega t) + l\dot{\alpha} \cos\alpha, \qquad \dot{y} = -r\omega \sin(\omega t) - l\dot{\alpha} \sin\alpha$$

we have

$$
\begin{aligned}
T' &= \tfrac{1}{2}m\left(r\omega\cos(\omega t) + l\dot{\alpha}\cos\alpha\right)^2 + \tfrac{1}{2}m\left(-r\omega\sin(\omega t) - l\dot{\alpha}\sin\alpha\right)^2 \\
&= \tfrac{1}{2}m\left[r^2\omega^2 + l^2\dot{\alpha}^2 + l\dot{\alpha}r\omega\cos(\omega t - \alpha)\right].
\end{aligned}
$$

The potential energy is associated to the gravitational force, thus

$$V = mgl(1 - \cos\alpha).$$

This defines the Lagrangian with

$$L = T' - V = \frac{1}{2}m\left[r^2\omega^2 + l^2\dot{\alpha}^2 + l\dot{\alpha}r\omega\cos(\omega t - \alpha)\right] - mgl(1 - \cos\alpha).$$

The application of the Lagrange formalism leads to the equation of motion

$$\alpha : \ddot{\alpha} - \frac{1}{2l}r\omega^2 \sin(\omega t - \alpha) + \frac{g}{l}\sin\alpha = 0.$$

2.4 Energy of Mechanical Systems

Let us assume that equality of the kinetic energy T and the kinetic coenergy $\dot{T}$. Then, the difference between the usual Lagrangian

$$L = T - V \tag{53}$$

and the energy $E = T + V$ is just the sign of V. Is there some general way to calculate E from the knowledge of L? We start with a definition

$$E = \sum_{i=1}^{n} \dot{q}_i \frac{\partial}{\partial \dot{q}_i} L - L \tag{54}$$

and prove, whether E satisfies the conditions to be an energy (state function) or not. The candidate E meets the relations

$$\tag{55}$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{n}\dot{q}_i\frac{\partial}{\partial\dot{q}_i}L - L\right)\\
&= \sum_{i=1}^{n}\dot{q}_i\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_i}L + \sum_{i=1}^{n}\frac{\partial}{\partial\dot{q}_i}L\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_i - \sum_{i=1}^{n}\dot{q}_i\frac{\partial}{\partial q_i}L - \sum_{i=1}^{n}\frac{\partial}{\partial\dot{q}_i}L\frac{\mathrm{d}}{\mathrm{d}t}\dot{q}_i - \frac{\partial}{\partial t}L\\
&= \sum_{i=1}^{n}\dot{q}_i\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_i}L - \frac{\partial}{\partial q_i}L\right) - \frac{\partial}{\partial t}L\\
&= \sum_{i=1}^{n}\dot{q}_iQ_i^{\mathrm{e}} - \frac{\partial}{\partial t}L
\end{aligned}
$$

If no external generalized forces Q_i^{e} exist and L is time independent than the relation

$$
\frac{\mathrm{d}}{\mathrm{d}t}E = 0 \tag{56}
$$

will be met. In this case, E is a constant of motion like the energy. However we have not established, whether E is in fact the energy $T + V$. We state without proof that if T is a homogenous quadratic function of $\dot{q}_i$, then E will be the energy [4]. Note, a function f is called homogenous quadratic, iff the condition

$$
f(az_1,\dots,az_k) = a^2 f(z_1,\dots,z_k) \tag{57}
$$

is satisfied. The conditions for T to be homogenous quadratic in $\dot{q}_i$ are:

- The potential V is independent of $\dot{q}_i$.
- The transformation from the Cartesian coordinates to the generalized coordinates is time independent.
- $L = T - V$ is time independent.

2.5 Legendre Transformations

The Lagrangian can be used to formulate the equations of motion of dynamical systems. In this section we discuss alternate state functions. The Hamiltonian or total energy can be obtained from the Lagrangian by a transformation of the variables. The generalized velocity $\dot{q}_i$ can be replaced by the associated variable p_i to get the Hamiltonian H which is a function of q_i and p_i. Using a Legendre transformation to define H gives

$$
-H = L - \sum_{i=1}^{n}p_i\dot{q}_i. \tag{58}
$$

Taking the total differential of H gives either – see equation (47)

$$
\tag{59}
$$

$$dH = \sum_{i=1}^{n}\left(-\frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}d\dot{q}_i + \dot{q}_i dp_i + p_i d\dot{q}_i\right) = \sum_{i=1}^{n}\left(-\frac{\partial L}{\partial q_i}dq_i + \dot{q}_i dp_i\right).$$

or

$$dH = \sum_{i=1}^{n}\left(\frac{\partial H}{\partial q_i}dq_i + \frac{\partial H}{\partial p_i}dp_i\right). \qquad (60)$$

In case of no external forces we have

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L - \frac{\partial}{\partial q_i}L = 0 \qquad \Rightarrow \qquad \frac{d}{dt}p_i = \frac{\partial}{\partial q_i}L \qquad (61)$$

and we can obtain

$$dH = \sum_{i=1}^{n}\left(-\dot{p}_i dq_i + \dot{q}_i dp_i\right) \qquad (62)$$

Finally, the comparison of (62) and (60) gives the Hamilton's equations of motion

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i, \qquad \frac{\partial H}{\partial p_i} = \dot{q}_i. \qquad (63)$$

The Legendre transformation offers the definition of other state functions, e.g.

$$L'(p_1,\dots,p_n,Q_1,\dots,Q_n) = H + \sum_{i=1}^{n}q_i Q_i \qquad (64)$$

or

$$H'(\dot{q}_1,\dots,\dot{q}_n,Q_1,\dots,Q_n) = L - \sum_{i=1}^{n}q_i Q_i. \qquad (65)$$

Usually these relations are not used in the modelling. Nevertheless the are of some theoretical interest. We have established that the Hamiltonian H and the total energy E are equal besides some less

restrictive conditions – compare equation (54) and equation (58). Now, the quantity $\dot{H}$ is defined to be the total coenergy

$$H' = T' + V' \qquad (66)$$

and it is called the co–Hamiltonian. Moreover, the quantity $\dot{L}$ is called the co–Lagrangian in defined as

$$L' = T - V'. \qquad (67)$$

Remark 11 *In this concept the definition of the potential energy has the general form*

$$V = - \int f \mathrm{d}q \tag{68}$$

whereas the definition of the potential coenergy has the general form

$$V' = - \int q \mathrm{d}f. \tag{69}$$

2.6 Case Studies

2.6.1 Atwood's Machine

Modelling and Simulation

2.6.2 Car and Beam

Modelling and Simulation

2.6.3 Double Pendulum

- frictionless: Modelling and Simulation
- with friction: Modelling and Simulation

2.6.4 Bead and Hoop

Modelling and Simulation

2.6.5 Ball on a Wheel

Modelling and Simulation

2.6.6 Two dimensional truck model

Modelling and Simulation

3 Electrical Systems

Section 2 has presented a modeling technique for mechanical systems by means of energy terms. For the coupling of electrical and mechanical systems, we have to extend this idea to electrical systems. We consider networks made up of resistors, capacitors, inductors, and sources. Resistors and sources are called the static terminals of the network. Capacitors and inductors are called the dynamic terminals. Later we discuss briefly the nature of these objects, called the branches of the circuit. At present, it suffices to consider them as devices with two terminals. The network is formed by connecting together various terminals. The connection points are called nodes. To find a mathematical description of the

network, we define a graph which corresponds to the networks. This graph $G = (\mathcal{N}, \mathcal{B})$ consists of the following data:

- A finite set N of points called nodes. The number of nodes is a.
- A finite set B of lines called branches. The number of branches is b. A branch (or port) has exactly two end points which must be nodes.

A current state of the network will be some vector $\mathbf{i}^T = [i_1, \ldots, i_b]$, where i_k represents the current flowing through the k–branch at a certain moment. Kirchhoff's current law states that the amount of current flowing into a node at a given moment is equal to the amount flowing out. For a node k, we get

$$\sum_l d_{kl} i_l = 0, \quad k \in \mathcal{B} . \tag{70}$$

The sum is taken over all branches and d_{kl} is defined as :

- d_{kl} = 1: if node k and branch l are connected and the direction of i_l to node k is positive
- d_{kl} = -1: if node k and branch l are connected and the direction of i_l to node k is negative
- d_{kl} = 0: otherwise

Next, a voltage state of the network is defined to be the vector $\mathbf{u}^T = [u_1, \ldots, u_b]$, where u_l represents the voltage drop across the l–th branch. Kirchhoff's voltage law states that there is a real function on the set of nodes, a voltage potential, so that

$$v_j d_{jl} + v_k d_{kl} = u_l \tag{71}$$

holds for each branch l. The power P of a network is a real function defined as

$$P = \sum_{k=1}^{b} i_k u_k \tag{72}$$

A current $\mathbf{i}^T = [i_1, \ldots, i_b] \in R^b$, and a voltage $\mathbf{u}^T = [u_1, \ldots, u_b] \in R^b$ are said to be admissible, if they obey the current law (70) and the voltage law (71). Telegen's theorem offers a very efficient way to characterize admissible currents and voltages. We state without proof [5]:

Theorem 1 (Telegen) *The relation*

$$\sum_{k=1}^{b} i_k u_k = 0 \tag{73}$$

is met for any admissible current $\mathbf{i}$ *and any admissible voltage* $\mathbf{u}$ *of a Kirchhoff network with graph G =* $(\mathcal{N}, \mathcal{B})$.

A direct consequence is that the power is zero for any admissible current $\mathbf{i}$ and any admissible voltage $\mathbf{u}$

of a Kirchhoff network.

3.1 Energy and Coenergy of Simple Devices

Now, we describe in mathematical terms the three different types of devices in the network; static terminals, capacitors, and inductors.

3.1.1 Static terminals

Each static terminal S imposes a relation

$$h(u,i) = 0 \tag{74}$$

on the current i and the voltage u of its branch. Typical examples are

- the linear resistor $u = Ri$,
- the voltage source $u = $ constant, i arbitrary and
- the current source $i = $ constant, u arbitrary.

Next, we introduce functions $P^{\mathrm{u}}(u)$, $P^{\mathrm{i}}(i)$ such that the relations

$$\frac{\partial}{\partial u}P^{\mathrm{u}} = i, \qquad \frac{\partial}{\partial i}P^{\mathrm{i}} = u, \qquad P^{\mathrm{u}} + P^{\mathrm{i}} = ui \tag{75}$$

are met. Note, the physical dimension of these quantities is Watt. If u and i has the same direction, that means $P^{\mathrm{u}} + P^{\mathrm{i}}$ is positive, then we will have dissipated power. Let us compare the definition

$$Q_j = -\frac{\partial}{\partial \dot{q}_j}P^{\mathrm{R}} \tag{76}$$

of the Rayleigh potential P^{R} of dissipative forces with the relations (75). For the dissipative forces of a mechanical system, we can define powers P^{u} and P^{i} in the way

$$\frac{\partial}{\partial \dot{q}_j}P^{\mathrm{u}} = -Q_j, \qquad \frac{\partial}{Q_j}P^{\mathrm{i}} = -\dot{q}_j, \qquad P^{\mathrm{u}} + P^{\mathrm{i}} = P \tag{77}$$

If $\dot{q}_j$ and Q_j are directed in opposite, then P^{u}, P^{i}, P will be positive functions. This is the usual convention of the direction of forces and velocities of dissipative forces. Taking the example

$$Q_D = -g\dot{q}, \tag{78}$$

then short calculations show

$$\tag{79}$$

$$\frac{\partial}{\partial \dot{q}} P^{\mathrm{u}} = -Q_D, \qquad \Rightarrow \qquad P^{\mathrm{u}} = \int\limits_0^{\dot{q}} g\dot{q}' \, \mathrm{d}\dot{q}' = g\frac{\dot{q}^2}{2} = P^{\mathrm{R}}$$

$$\frac{\partial}{\partial Q_D} P^{\mathrm{i}} = -\dot{q}, \qquad \Rightarrow \qquad P^{\mathrm{i}} = \int\limits_0^{Q_D} \frac{Q_D'}{g} \, \mathrm{d}Q_D' = \frac{Q_D^2}{2g}$$

and

$$P^{\mathrm{u}} + P^{\mathrm{i}} = g\frac{\dot{q}^2}{2} + \frac{Q_D^2}{2g} = g\frac{\dot{q}^2}{2} + \frac{g^2\dot{q}^2}{2g} = g\dot{q}^2 = -Q_D\dot{q}. \tag{80}$$

Relations ([75](#)) and ([77](#)) look very similar. The difference is just the sign. If u and i are directed in the same direction, then the power $P^{\mathrm{u}} + P^{\mathrm{i}} > 0$. This is a usual convention in electrical networks. For this similarity, we call the powers of the electrical network generalized potentials, too. For example, a linear resistance satisfies $u = Ri$, which leads to

$$P^{\mathrm{u}} = \int\limits_0^u i\left(u'\right) \mathrm{d}u' = \int\limits_0^u \frac{u'}{R} \mathrm{d}u' = \frac{u^2}{2R}$$

$$P^{\mathrm{i}} = \int\limits_0^i u\left(i'\right) \mathrm{d}i' = \int\limits_0^i Ri' \, \mathrm{d}i' = \frac{Ri^2}{2}. \tag{81}$$

The resistor is a dissipative element. Hence i and u of a resistor have the same direction.
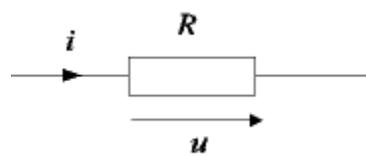
Figure 5: Resistor.

The relations ([81](#)) allow a simple interpretation of the quantities $P^{\mathrm{u}}(u)$, $P^{\mathrm{i}}(i)$.

Remark 12 *Figure ([6](#)) shows a nonlinear resistor law. We can deduce that P^{i} and P^{u} can be interpreted as areas above and beyond the curve.*
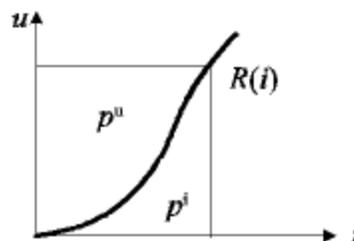
Figure 6: Interpretation

A voltage source satisfies $u = u_0$, which leads to

$$P^{\mathrm{i}} = \int_0^i u\left(i'\right) \mathrm{d}i' = -u_0 i, \qquad P^{\mathrm{u}} = 0. \tag{82}$$

Note, u and i don't have the same direction. Hence $P^{\mathrm{i}} < 0$.
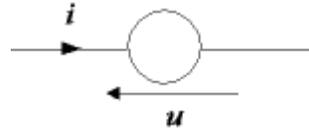
Figure 7: Voltage source.

Analogous, a current source satisfies $i = i_0$, which leads to

$$P^{\mathrm{u}} = \int_0^u i\left(u'\right) \mathrm{d}u' = -i_0 u, \qquad P^{\mathrm{i}} = 0. \tag{83}$$
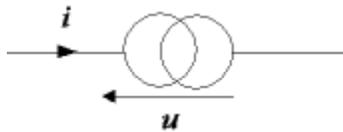
Figure 8: Current source.

3.1.2 Dynamic Devices

An inductor or capacitor does not impose conditions directly on the state, but defines how the state in that branch changes in time. In particular a capacitor and an inductor link the charge q with the voltage u and the flux ψ with the current i by the nonlinear relations

$$q = f\left(u\right), \qquad \psi = g\left(i\right) . \tag{84}$$

Their dynamics is given by the relations

$$\frac{\mathrm{d}}{\mathrm{d}t} q = i, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \psi = u \tag{85}$$

for a capacitor and an inductor, respectively. The combination of (84) and (85) immediately leads to

$$\tag{86}$$

$$\frac{\partial}{\partial u} f \frac{d}{dt} u = i, \qquad \frac{\partial}{\partial i} g \frac{d}{dt} i = u .$$

Remark 13 *A linear capacitor is given by* $q = f(u) = Cu$, *which leads to the relation*

$$\frac{d}{dt} q = C \frac{d}{dt} u = i . \tag{87}$$

Remark 14 *A linear inductor is given by* $\psi = g(i) = Li$, *which leads to the relation*

$$\frac{d}{dt} \psi = L \frac{d}{dt} i = u . \tag{88}$$

Now, let us introduce the energy W^q of the capacitor and the energy W^ψ of the inductor. We assume that there exist functions W^q, W^ψ such that

$$\frac{\partial}{\partial q} W^q = u, \qquad \frac{\partial}{\partial \psi} W^\psi = i \tag{89}$$

or equivalently

$$W^q = \int_0^q u\left(q'\right) dq', \qquad W^\psi = \int_0^\psi i\left(\psi'\right) d\psi' \tag{90}$$

is met. The prime denotes the variable of integration. One gets

$$\frac{d}{dt} W^q = \frac{\partial}{\partial q} W^q \frac{d}{dt} q = ui \tag{91}$$

and

$$\frac{d}{dt} W^\psi = \frac{\partial}{\partial \psi} W^\psi \frac{d}{dt} \psi = iu. \tag{92}$$

Remark 15 *Here, the coordinates* q *and* ψ *are considered as independent variables.*

Remark 16 *The relations*

$$\frac{d}{dt} q = i, \qquad \frac{d}{dt} \psi = u \tag{93}$$

motivate the interpretation of the charge q *and the flux linkage* ψ *as "position" coordinates. The*

associated velocities are the currents and the voltages. Such an interpretation is sometimes used in modeling, see [?].

In a similar way, we can define dual objects W^u, W^i with

$$\frac{\partial}{\partial u}W^u = q, \qquad \frac{\partial}{\partial i}W^i = \psi, \tag{94}$$

which are called electrical coenergies. They meet the relations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(W^q + W^u\right) = \frac{\partial}{\partial q}W^q\frac{\mathrm{d}}{\mathrm{d}t}q + \frac{\partial}{\partial u}W^u\frac{\mathrm{d}}{\mathrm{d}t}u = u\frac{\mathrm{d}}{\mathrm{d}t}q + q\frac{\mathrm{d}}{\mathrm{d}t}u = \frac{\mathrm{d}}{\mathrm{d}t}\left(u\,q\right) \tag{95}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(W^\psi + W^i\right) = \frac{\partial}{\partial \psi}W^\psi\frac{\mathrm{d}}{\mathrm{d}t}\psi + \frac{\partial}{\partial i}W^i\frac{\mathrm{d}}{\mathrm{d}t}i = i\frac{\mathrm{d}}{\mathrm{d}t}\psi + \psi\frac{\mathrm{d}}{\mathrm{d}t}i = \frac{\mathrm{d}}{\mathrm{d}t}\left(i\psi\right) \tag{96}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial u}W^u\right) = i, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial i}W^i\right) = u. \tag{97}$$

Remark 17 *Here, the coordinates u and i are considered as independent variables.*

For example, a linear capacitor satisfies $q = Cu$, which leads to the energy

$$W^q = \int_0^q u\left(q'\right)\mathrm{d}q' = \int_0^q \frac{q'}{C}\mathrm{d}q' = \frac{q^2}{2C} \tag{98}$$

and the coenergy

$$W^u = \int_0^u q\left(u'\right)\mathrm{d}u' = \int_0^u Cu'\,\mathrm{d}u' = \frac{Cu^2}{2}. \tag{99}$$
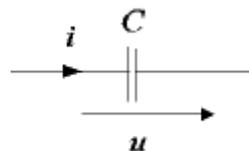
Figure 9: Capacitor.

For linear capacitors, the energy and the coenergy are equal. This fact does not hold for nonlinear

devices. Analogous, a linear inductor satisfies $\psi = Li$, which leads to the energy

$$W^\psi = \int_0^\psi i\left(\psi'\right) d\psi' = \int_0^\psi \frac{\psi'}{L} d\psi' = \frac{\psi^2}{2L} \tag{100}$$

and the coenergy

$$W^i = \int_0^i \psi\left(i'\right) di' = \int_0^i Li' \, di' = \frac{Li^2}{2}. \tag{101}$$
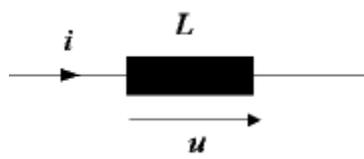
Figure 10: Inductor.

3.1.3 Example

Simulation models of MOS–FET devices take into account that the charge storage can be described by voltage depending capacitors

$$q = C\left(u\right) u.$$

If the voltage u drop across the bulk–drain branch is positive, then we have the relation for the capacitor

$$C = C_0 \left(1 + \frac{u}{\Phi}\right) \qquad C_0 = 2.4 \cdot 10^{-14}\,[\mathrm{F}]\,, \Phi = 0.87\,[\mathrm{V}]\,.$$

Then, the energy W^q and the coenergy W^u are given as:

$$W^q = \int_0^u q\left(z\right) dz = C_0 u^2 \left(\frac{1}{2} + \frac{1}{3}\frac{u}{\Phi}\right)$$

$$W^u = -\frac{1}{24}\left(-C_0\phi + \sqrt{C_0\phi\left(C_0\phi + 4q\right)}\right)\left(-8q - C_0\phi + \sqrt{C_0\phi\left(C_0\phi + 4q\right)}\right)/C_0$$

3.2 Equations of Motion of Simple Networks

As mentioned above, a simple network is built up by three types of terminals. Therefore, the set B can be subdivided into 3 disjoint sets L, C and S of subsets of B in such a way, that the sets L, C, S contain the inductors, the capacitors and the static terminals, respectively, and the relation $B = L \cup C \cup S$ is met.

Remark 18 *For our proposed type of electrical networks, the current in the inductors and the voltage*

drops along the capacitors, via Kirchhoff's laws and the laws of the static terminals, determine the currents and the voltages in all the branches. We call such networks simple. If capacitors are connected in parallel or inductances are connected in series, then such devices should be combined to one single device.

All voltages $(\mathbf{u}_C, \mathbf{u}_L, \mathbf{u}_S)$ are expressed as functions of the voltages $\mathbf{u}_C{}^T = [u_{C,1}, \cdots, u_{C,n_C}]$ of the capacitors and the currents $\mathbf{i}_L{}^T = [i_{L,1}, \cdots, i_{L,n_L}]$ of the inductors. In a similar way, we express all currents as functions of $\mathbf{u}_c$ and $\mathbf{i}_L$. This gives two maps

$$\varphi : (\mathbf{u}_C, \mathbf{i}_L) \rightarrow (\mathbf{u}_C, \mathbf{u}_L, \mathbf{u}_S), \qquad \psi : (\mathbf{u}_C, \mathbf{i}_L) \rightarrow (\mathbf{i}_C, \mathbf{i}_L, \mathbf{i}_S) \tag{102}$$

Without saying, $(\mathbf{u}_C, \mathbf{u}_L, \mathbf{u}_S)$, and $(\mathbf{i}_C, \mathbf{i}_L, \mathbf{i}_S)$ meet Telegen's theorem. The total coenergy of the network is defined by

$$W^{\text{el}}(\mathbf{u}_C, \mathbf{i}_L) = \sum_{\mathcal{C}} W^{\mathrm{u}}(\mathbf{u}_C) + \sum_{\mathcal{L}} W^{\mathrm{i}}(\mathbf{i}_L). \tag{103}$$

We will follow the convention that sums over C or L means summation over all capacitor or inductor branches. Further, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u_{C,k}} W^{\text{el}} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial}{\partial u_{C,k}} \left(\sum_{\mathcal{C}} W^{\mathrm{u}}(\mathbf{u}_C) \right) + \frac{\partial}{\partial u_{C,k}} \left(\sum_{\mathcal{L}} W^{\mathrm{i}}(\mathbf{i}_L) \right) \right) = \frac{\mathrm{d}}{\mathrm{d}t} q_k = i_k \tag{104}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,l}} W^{\text{el}} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial}{\partial i_{L,l}} \left(\sum_{\mathcal{C}} W^{\mathrm{u}}(\mathbf{u}_C) \right) + \frac{\partial}{\partial i_{L,l}} \left(\sum_{\mathcal{L}} W^{\mathrm{i}}(\mathbf{i}_L) \right) \right) = \frac{\mathrm{d}}{\mathrm{d}t} \psi_l = u_l \tag{105}$$

with $k \in C$, $l \in L$. We define the quantities

$$P^{\mathrm{C}}(\mathbf{u}_C, \mathbf{i}_L) = \sum_{\mathcal{C}} ui, \qquad P^{\mathrm{L}}(\mathbf{u}_C, \mathbf{i}_L) = \sum_{\mathcal{L}} ui \tag{106}$$

and

$$P^{\mathrm{u}}(\mathbf{u}_C, \mathbf{i}_L) = \sum_{S} P^{\mathrm{u}}(u), \qquad P^{\mathrm{i}}(\mathbf{u}_C, \mathbf{i}_L) = \sum_{S} P^{\mathrm{i}}(i) \tag{107}$$

which are powers. They are needed for the derivation of the equations of motion. Telegen's theorem can be rewritten as

$$\sum_{k=1}^{b} i_k u_k = \sum_{\mathcal{C}} ui + \sum_{\mathcal{L}} ui + \sum_{S} ui = P^{\mathrm{C}} + P^{\mathrm{L}} + P^{\mathrm{u}} + P^{\mathrm{i}} = 0. \tag{108}$$

Please remind that sums over C or L means summation over all capacitor or inductor branches. At a

given time t_0 the circuit is in a particular current $[i_1, \ldots, i_b] \in \mathbf{R}^b$ – voltage $[u_1, \ldots, u_b] \in \mathbf{R}^b$ state. In this way, a curve is obtained, depending on the initial state of the circuit. The components i_k, u_k, $k \in$ B of this curve must satisfy the conditions imposed by Kirchhoff's laws and the static terminal laws. In addition, at a given time the components du_k/dt and di_k/dt of the tangent vectors of the curve must satisfy the relations (104) and (105). A curve satisfying these conditions is called a trajectory. The coordinates i_k, u_k, $k \in$ B has a property in common with the coordinates $\mathbf{x}_i$ of a specific particle of a mechanical system. They describe the system completely but in general they are not independent from each other due to the restrictions. For this reason, we have introduced the generalized coordinates q_j. In the present case, the currents through the inductors and the voltage drops along the capacitors play the analogous role. As supposed in remark (18), they determine the system completely. Moreover, they are independent from each other.

Next, we are going to state this set of equations of motion for simple electrical networks. Let $\mathbf{u}$ be any admissible voltage. Then, the time derivative $d\mathbf{u}/dt$ fulfills the Kirchhoff voltage law and is an admissible voltage, too. Hence, Telegen's theorem tells us

$$\sum_{k=1}^{b} i_k \frac{du_k}{dt} = 0 \tag{109}$$

for any admissible current $\mathbf{i}$. We rewrite this as

$$\sum_{\mathcal{C}} i \frac{du}{dt} + \sum_{\mathcal{L}} i \frac{du}{dt} + \sum_{\mathcal{S}} i \frac{du}{dt} = 0. \tag{110}$$

From the Leibnitz rule we get

$$\sum_{\mathcal{L}} i \frac{du}{dt} = \frac{d}{dt} \left(\sum_{\mathcal{L}} iu \right) - \sum_{\mathcal{L}} u \frac{di}{dt} = -\sum_{\mathcal{C}} i \frac{du}{dt} - \sum_{\mathcal{S}} i \frac{du}{dt}. \tag{111}$$

This leads to

$$\frac{d}{dt} \left(\sum_{\mathcal{L}} iu \right) - \sum_{\mathcal{L}} u \frac{di}{dt} + \sum_{\mathcal{C}} i \frac{du}{dt} + \sum_{\mathcal{S}} i \frac{du}{dt} = 0 \tag{112}$$

and – see equations (106) and (107) –

$$\frac{d}{dt} P^{\mathrm{L}} + \frac{d}{dt} P^{\mathrm{u}} - \sum_{\mathcal{L}} u \frac{di}{dt} + \sum_{\mathcal{C}} i \frac{du}{dt} = 0 . \tag{113}$$

P^{L} and P^{u} are functions of the currents through the inductors and the voltage drops along the capacitors. Hence, the chain rule gives

$$\sum_{\mathcal{C}} \frac{\partial}{\partial u} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) \frac{du}{dt} + \sum_{\mathcal{L}} \frac{\partial}{\partial i} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) \frac{di}{dt} - \sum_{\mathcal{L}} u \frac{di}{dt} + \sum_{\mathcal{C}} i \frac{du}{dt} = 0 \tag{114}$$

and

$$\sum_{\mathcal{C}} \left(\frac{\partial}{\partial u} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) + i \right) \frac{\mathrm{d}u}{\mathrm{d}t} + \sum_{\mathcal{L}} \left(\frac{\partial}{\partial i} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) - u \right) \frac{\mathrm{d}i}{\mathrm{d}t} = 0. \qquad (115)$$

Since $\mathrm{d}u_k/\mathrm{d}t$ and $\mathrm{d}i_k/\mathrm{d}t$ can take any value,

$$\frac{\partial}{\partial u_k} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) + i_k = 0, \qquad \frac{\partial}{\partial i_k} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) - u_k = 0. \qquad (116)$$

Taking into account

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u_{C,k}} W^{\mathrm{el}} = i_{C,k}, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,k}} W^{\mathrm{el}} = u_{L,k} \qquad (117)$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u_{C,k}} W^{\mathrm{el}} + \frac{\partial}{\partial u_{C,k}} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) = 0, \qquad k \in \mathcal{C} \qquad (118)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,k}} W^{\mathrm{el}} - \frac{\partial}{\partial i_{L,k}} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) = 0 \qquad k \in \mathcal{L}. \qquad (119)$$

Using the equation (108) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,k}} W^{\mathrm{el}} + \frac{\partial}{\partial i_{L,k}} \left(P^{\mathrm{C}} + P^{\mathrm{i}} \right) = 0, \qquad k \in \mathcal{L}. \qquad (120)$$

The equations (118) and (120) state the set of equations of motions for simple electrical networks – see [5].

Remark 19 *The right hand sides of the differential equations (118) and (120) are functions of all $u_{C,k}$, $i_{L,k}$. This fact coincides with remark (18).*

Remark 20 *The set of the differential equations (118) and (120) is a set of first–order differential equations.*

3.2.1 Example I

Suppose we have the simple electrical network given in figure 11.
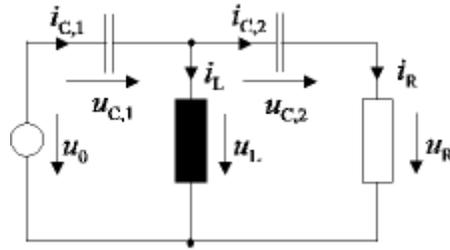
Figure 11: A simple electrical network.

First we will calculate the maps φ and ψ as introduced in ([106](#)) and ([107](#)). This means nothing that all voltages and currents must be expressed as functions of the voltages across the capacitors and currents through the inductances. From the voltage and current laws it follows

$$u_0 = u_{C,1} + u_L$$
$$u_L = u_{C,2} + u_R$$
$$i_{C,1} = i_{C,2} + i_L$$
$$i_{C,2} = i_R$$

and

$$u_L = u_0 - u_{C,1}$$
$$u_R = u_L - u_{C,2} = u_0 - u_{C,1} - u_{C,2}$$
$$i_R = \frac{u_R}{R} = \frac{u_0 - u_{C,1} - u_{C,2}}{R}$$
$$i_{C,1} = i_{C,2} + i_L = \frac{u_0 - u_{C,1} - u_{C,2}}{R} + i_L$$
$$i_{C,2} = i_R = \frac{u_0 - u_{C,1} - u_{C,2}}{R}.$$

Now, the power $P^C(\mathbf{u}_C, \mathbf{i}_L)$ follows with

$$P^C(\mathbf{u}_C, \mathbf{i}_L) = u_{C,1} i_{C,1} + u_{C,2} i_{C,2}$$
$$= u_{C,1} \left(\frac{u_0 - u_{C,1} - u_{C,2}}{R} + i_L \right) + u_{C,2} \left(\frac{u_0 - u_{C,1} - u_{C,2}}{R} \right)$$

and $P^u(\mathbf{u}_C, \mathbf{i}_L)$ is given by

$$P^u(\mathbf{u}_C, \mathbf{i}_L) = \frac{u_R^2}{2R} + 0 = \frac{(u_0 - u_{C,1} - u_{C,2})^2}{2R}.$$

The calculation of the total coenergy of the network leads to

$$W^{el}(\mathbf{u}_C, \mathbf{i}_L) = \sum_{\mathcal{C}} W^u(\mathbf{u}_C) + \sum_{\mathcal{L}} W^i(\mathbf{i}_L) = \frac{C_1 u_{C,1}^2}{2} + \frac{C_2 u_{C,2}^2}{2} + \frac{L i_L^2}{2}.$$

Moreover we have

$$P^{\mathrm{L}}\left(\mathbf{u}_C,\mathbf{i}_L\right) = \left(u_0 - u_{C,1}\right)i_L$$

and

$$P^{\mathrm{i}}\left(\mathbf{u}_C,\mathbf{i}_L\right) = \frac{R}{2}\left(\frac{u_0 - u_{C,1} - u_{C,2}}{R}\right)^2 - u_0\left(\frac{u_0 - u_{C,1} - u_{C,2}}{R} + i_L\right).$$

The evaluation of

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial u_{C,k}}W^{\mathrm{el}} + \frac{\partial}{\partial u_{C,k}}\left(P^{\mathrm{L}} + P^{\mathrm{u}}\right) = 0, \qquad k \in \mathcal{C}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial u_{C,1}}W^{\mathrm{el}} + \frac{\partial}{\partial u_{C,1}}\left(P^{\mathrm{L}} + P^{\mathrm{u}}\right) = C_1\frac{\mathrm{d}}{\mathrm{d}t}u_{C,1} + \frac{u_{C,1} + u_{C,2}}{R} - \frac{u_0}{R} - i_L = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial u_{C,2}}W^{\mathrm{el}} + \frac{\partial}{\partial u_{C,2}}\left(p^{\mathrm{L}} + p^{\mathrm{u}}\right) = C_2\frac{\mathrm{d}}{\mathrm{d}t}u_{C,2} + \frac{u_{C,1} + u_{C,2}}{R} - \frac{u_0}{R} = 0.$$

The evaluation of

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial i_{L,k}}W^{\mathrm{el}} + \frac{\partial}{\partial i_{L,k}}\left(P^{\mathrm{C}} + P^{\mathrm{i}}\right) = 0, \qquad k \in \mathcal{L}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial i_L}W^{\mathrm{el}} + \frac{\partial}{\partial i_L}\left(P^{\mathrm{C}} + P^{\mathrm{i}}\right) = L\frac{\mathrm{d}}{\mathrm{d}t}i_L + u_{C,1} - u_0 = 0.$$

3.2.2 Example II, Cuk–Converter

The Cuk–Converter is a special case of a dc–dc converter, which is widely used in switch–mode dc power supplies and dc motor drive applications. As shown in figure (12), often the input of such converters is an unregulated dc voltage U_e. Switch–mode dc–dc converters are used to convert the unregulated dc input into a controlled dc output $u_{C,2}$ at a desired voltage level. The output voltage $u_{C,2}$ may be higher or lower than the input voltage.
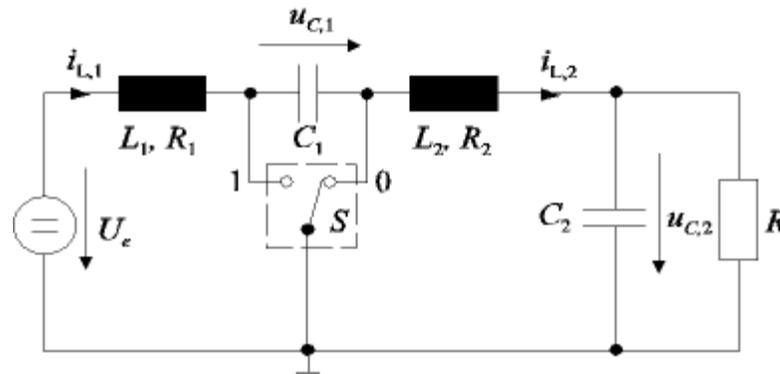
Figure 12: Cuk–Converter.

For the analysis, the switch is treated as being ideal and the capacitive elements C_1, C_2 have no losses. The losses in the inductances L_1, L_2 are modelled by resistors R_1, R_2. The dc input voltage U_e to the converter is assumed to have zero internal impedance. The output is assumed to supply a load that can be represented by an equivalent resistor R. The control input u is called the duty ratio u, $0 \le u \le 1$ and this quantity specifies the ratio of the duration of the switch S in position 1 to the fixed modulation period T.

If the switch S is in position 0, then the circuit can be divided in two parts as shown in figure ([13](#)).
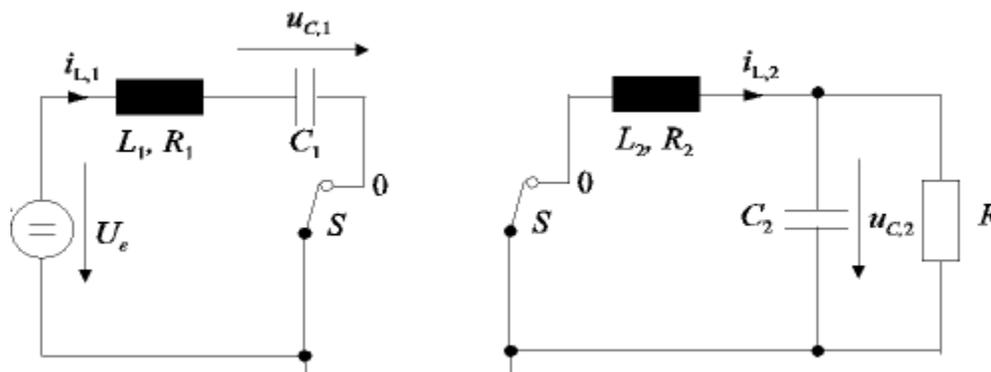
Figure 13: The switch S is in position 0.

First, we will calculate the maps φ and ψ as introduced in ([106](#)) and ([107](#)). From the voltage and current laws it follows

$$U_e = u_{C,1} + u_{L,1} + u_{R,1}$$
$$0 = u_{C,2} + u_{L,2} + u_{R,2}$$
$$i_{L,2} = i_{R,2} = i_R + i_{C,2}$$

and

$$u_{L,1} = U_e - u_{C,1} - i_{L,1}R_1$$

$$u_{L,2} = -u_{C,2} - i_{L,2}R_2$$

$$i_{C,2} = -\frac{u_{C,2}}{R} + i_{L,2}\ .$$

Trivial equations are not displayed. The proposed modeling technique requires the quantities

$$P^C\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{C}} ui = u_{C,1}i_{L,1} + u_{C,2}\left(-\frac{u_{C,2}}{R} + i_{L,2}\right)$$

$$P^L\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{L}} ui = \left(U_e - u_{C,1} - i_{L,1}R_1\right)i_{L,1} + \left(-u_{C,2} - i_{L,2}R_2\right)i_{L,2}$$

$$P^u\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{S}} P^u\left(u\right) = \frac{i_{L,1}^2 R_1}{2} + \frac{i_{L,2}^2 R_2}{2} + \frac{u_{C,2}^2}{2R}$$

$$P^i\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{S}} P^i\left(u\right) = \frac{i_{L,1}^2 R_1}{2} + \frac{i_{L,2}^2 R_1}{2} + \frac{u_{C,2}^2}{2R} - U_e i_{L,1}$$

and the total coenergy of the network

$$W^{el}\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{C}} W^u\left(\mathbf{u}_C\right) + \sum_{\mathcal{L}} W^i\left(\mathbf{i}_L\right) = \frac{C_1 u_{C,1}^2}{2} + \frac{C_2 u_{C,2}^2}{2} + \frac{L_1 i_{L,1}^2}{2} + \frac{L_2 i_{L,2}^2}{2}\ .$$

The evaluation of

$$\frac{d}{dt}\frac{\partial}{\partial u_{C,k}}W^{el} + \frac{\partial}{\partial u_{C,k}}\left(P^L + P^u\right) = 0, \qquad k \in \mathcal{C}$$

leads to

$$\frac{d}{dt}\frac{\partial}{\partial u_{C,1}}W^{el} + \frac{\partial}{\partial u_{C,1}}\left(P^L + P^u\right) = C_1\frac{d}{dt}u_{C,1} - i_{L,1} = 0$$

and

$$\frac{d}{dt}\frac{\partial}{\partial u_{C,2}}W^{el} + \frac{\partial}{\partial u_{C,2}}\left(P^L + P^u\right) = C_2\frac{d}{dt}u_{C,2} - i_{L,2} + \frac{u_{C,2}}{R} = 0\ .$$

The evaluation of

$$\frac{d}{dt}\frac{\partial}{\partial i_{L,k}}W^{el} + \frac{\partial}{\partial i_{L,k}}\left(P^C + P^i\right) = 0, \qquad k \in \mathcal{L}$$

leads to

$$\frac{d}{dt}\frac{\partial}{\partial i_{L,1}}W^{el} + \frac{\partial}{\partial i_{L,1}}\left(P^C + P^i\right) = L_1\frac{d}{dt}i_{L,1} + u_{C,1} + i_{L,1}R_1 - U_e = 0$$

and

$$\frac{d}{dt}\frac{\partial}{\partial i_{L,2}}W^{el} + \frac{\partial}{\partial i_{L,2}}\left(P^C + P^i\right) = L_2\frac{d}{dt}i_{L,2} + u_{C,2} + i_{L,2}R_2 = 0 \,.$$

If the switch S is in position 1, than the circuit can be divided in two parts as shown in figure ([13](#)).
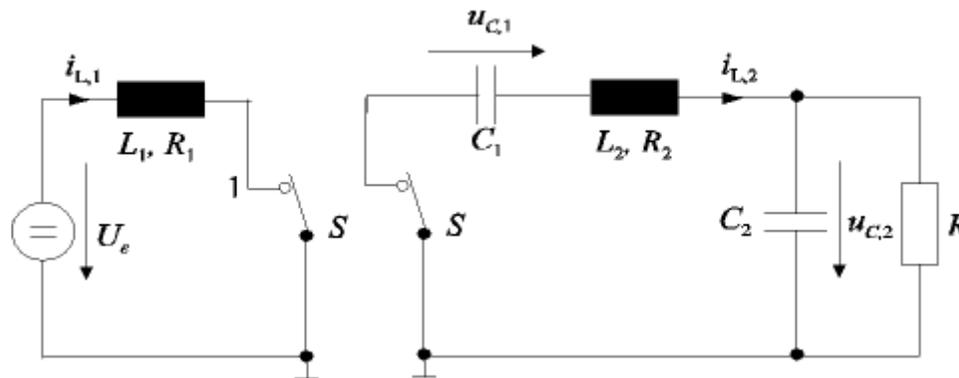
Figure 14: The switch is in position 1.

First we will calculate the maps φ and ψ as introduced in ([106](#)) and ([107](#)), again. From the voltage and current laws it follows

$$U_e = u_{L,1} + u_{R,1}$$
$$0 = u_{C,1} + u_{C,2} + u_{L,2} + u_{R,2}$$
$$i_{L,2} = i_{R,2} = i_{C,2} = i_R + i_{C,2}$$

and

$$u_{L,1} = U_e - i_{L,1}R_1$$
$$u_{L,2} = -u_{C,1} - u_{C,2} - i_{L,2}R_2$$
$$i_{C,2} = -\frac{u_{C,2}}{R} + i_{L,2}\,.$$

Trivial equations are not displayed. The proposed modeling technique requires the quantities

$$P^C\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{C}} ui = u_{C,1}i_{L,2} - u_{C,2}\left(-\frac{u_{C,2}}{R} + i_{L,2}\right)$$

$$P^L\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{L}} ui = \left(U_e - i_{L,1}R_1\right)i_{L,1} + \left(-u_{C,1} - u_{C,2} - i_{L,2}R_2\right)i_{L,2}$$

$$P^u\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{S}} P^u\left(u\right) = \frac{i_{L,1}^2 R_1}{2} + \frac{i_{L,2}^2 R_1}{2} + \frac{u_{C,2}^2}{2R}$$

$$P^i\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{S}} P^i\left(u\right) = \frac{i_{L,1}^2 R_1}{2} + \frac{i_{L,2}^2 R_1}{2} + \frac{u_{C,2}^2}{2R} - U_e i_{L,1}$$

and the total coenergy of the network

$$W^{\text{el}}\left(\mathbf{u}_C, \mathbf{i}_L\right) = \sum_{\mathcal{C}} W^{\text{u}}\left(\mathbf{u}_C\right) + \sum_{\mathcal{L}} W^{\text{i}}\left(\mathbf{i}_L\right) = \frac{C_1 u_{C,1}^2}{2} + \frac{C_2 u_{C,2}^2}{2} + \frac{L_1 i_{L,1}^2}{2} + \frac{L_2 i_{L,2}^2}{2} \, .$$

The evaluation of

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u_{C,k}} W^{\text{el}} + \frac{\partial}{\partial u_{C,k}} \left(P^{\text{L}} + P^{\text{u}}\right) = 0, \qquad k \in \mathcal{C}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u_{C,1}} W^{\text{el}} + \frac{\partial}{\partial u_{C,1}} \left(P^{\text{L}} + P^{\text{u}}\right) = C_1 \frac{\mathrm{d}}{\mathrm{d}t} u_{C,1} - i_{L,2} = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u_{C,2}} W^{\text{el}} + \frac{\partial}{\partial u_{C,2}} \left(P^{\text{L}} + P^{\text{u}}\right) = C_2 \frac{\mathrm{d}}{\mathrm{d}t} u_{C,2} - i_{L,2} + \frac{u_{C,2}}{R} = 0 \, .$$

The evaluation of

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,k}} W^{\text{el}} + \frac{\partial}{\partial i_{L,k}} \left(P^{\text{C}} + P^{\text{i}}\right) = 0, \qquad k \in \mathcal{L}$$

leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,1}} W^{\text{el}} + \frac{\partial}{\partial i_{L,1}} \left(P^{\text{C}} + P^{\text{i}}\right) = L_1 \frac{\mathrm{d}}{\mathrm{d}t} i_{L,1} + i_{L,1} R_1 - U_e = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,2}} W^{\text{el}} + \frac{\partial}{\partial i_{L,2}} \left(P^{\text{C}} + P^{\text{i}}\right) = L_2 \frac{\mathrm{d}}{\mathrm{d}t} i_{L,2} + u_{C,1} + u_{C,2} + i_{L,2} R_2 = 0 \, .$$

Generally a PWM controlled converter like the Čuk converter is described by two systems of differential equations of the form

$$\begin{aligned}
\dot{\mathbf{x}}_1 &= \mathbf{a}_1(\mathbf{x}_1) & t \in (iT, (i+u)T] & \qquad S \text{ in } 1 \\
\dot{\mathbf{x}}_2 &= \mathbf{a}_2(\mathbf{x}_2) & t \in ((i+u)T, (i+1)T] & \qquad S \text{ in } 0
\end{aligned} \tag{121}$$

for $i = 0, 1, \ldots$ with the smooth vector fields $\mathbf{a}_1$, $\mathbf{a}_2$ and the duty ratio u, $0 \leq u \leq 1$. For the Čuk converter we have

$$\mathbf{a}_1\left(\mathbf{x}\right) = \begin{bmatrix} \dfrac{U_e - i_{L,1}R_1}{L_1} \\[2mm] -\dfrac{u_{C,2} + u_{C,1} + i_{L,2}R_2}{L_2} \\[2mm] \dfrac{i_{L,2}}{C_1} \\[2mm] \dfrac{i_{L,2} - \dfrac{u_{C,2}}{R}}{C_2} \end{bmatrix} \quad \text{and} \quad \mathbf{a}_2\left(\mathbf{x}\right) = \begin{bmatrix} \dfrac{U_e - i_{L,1}R_1 - u_{C,1}}{L_1} \\[2mm] -\dfrac{i_{L,2}R_2 + u_{C,2}}{L_2} \\[2mm] \dfrac{i_{L,1}}{C_1} \\[2mm] \dfrac{i_{L,2} - \dfrac{u_{C,2}}{R}}{C_2} \end{bmatrix}$$

with the state $x^T = \begin{bmatrix} i_{L,1}, & i_{L,2}, & u_{C,1}, & u_{C,2} \end{bmatrix}$. The duty ratio thus specifies the ratio of the duration of the switch S in position 1 to the fixed modulation period T (see Fig. (15)).
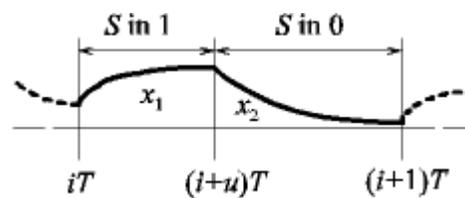
Figure 15: Duty ratio of a PWM controlled system.

From the theory of differential equations it is a well known fact that the state variables of a system $\dot{x} = f(x, u)$ with piecewise continuous inputs u are continuous [5]. Therefore, the two systems of (121) are connected by the conditions

$$\begin{aligned} \mathbf{x}_1(iT) &= \mathbf{x}_2(iT) \\ \mathbf{x}_1((i+u)T) &= \mathbf{x}_2((i+u)T) . \end{aligned}$$

Under the assumption that the switching frequency is much higher than the natural frequencies of the converter system and the switches are realized with common power semiconductor devices, we can derive the so called average model for the PWM controlled converter (121) in the form

$$\dot{\mathbf{x}} = \mathbf{a}_2\left(\mathbf{x}\right) + \left(\mathbf{a}_1\left(\mathbf{x}\right) - \mathbf{a}_2\left(\mathbf{x}\right)\right)u \tag{122}$$

with the average state vector $\mathbf{x}$ and the duty ratio u – see [4]. Hence, the average model for the Čuk converter reads as

$$\dot{\mathbf{x}} = \begin{bmatrix} \dfrac{U_e - x_1 R_1 - x_3}{L_1} \\[2mm] -\dfrac{x_2 R_2 + x_4}{L_2} \\[2mm] \dfrac{x_1}{C_1} \\[2mm] \dfrac{x_2 - \dfrac{x_4}{R}}{C_2} \end{bmatrix} + \begin{bmatrix} \dfrac{x_3}{L_1} \\[2mm] -\dfrac{x_3}{L_2} \\[2mm] \dfrac{x_2 - x_1}{C_1} \\[2mm] 0 \end{bmatrix} u$$

with the state $\mathbf{x}^T = [i_{L,1},\ i_{L,2},\ u_{C,1},\ u_{C,2}]$.

   More information

3.3 The Energy of Electrical Systems

The energy of all capacitors and inductors of an electrical network is given by

$$W^{\mathrm{el}}(\mathbf{u}_C, \mathbf{i}_L) = \sum_{\mathcal{C}} W^{\mathrm{u}}(\mathbf{u}_C) + \sum_{\mathcal{L}} W^{\mathrm{i}}(\mathbf{i}_L).$$

(123)

Using Telegen's theorem we get

$$\sum_{\mathcal{C}} ui + \sum_{\mathcal{L}} ui + \sum_{\mathcal{S}} ui = P^C + P^L + P^{\mathrm{u}} + P^{\mathrm{i}}$$

(124)

$$\sum_{\mathcal{C}} ui + \sum_{\mathcal{L}} ui + \sum_{\mathcal{S}} ui = \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\mathcal{C}} W^{\mathrm{u}} + \sum_{\mathcal{L}} W^{\mathrm{i}}\right) + P^{\mathrm{u}} + P^{\mathrm{i}} = 0$$

(125)

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{\mathcal{C}} W^{\mathrm{u}} + \sum_{\mathcal{L}} W^{\mathrm{i}}\right) = -P^{\mathrm{u}} - P^{\mathrm{i}} = -P^{\mathrm{S}}.$$

(126)

These relations may be interpreted in the way that in a circuit the energy in the inductors and capacitors varies according to the power dissipated in the resistors and supplied by the sources.

4 Electromechanical Systems

4.1 Introduction

The overall idea of the coupling of electrical and mechanical systems will be described by the following example. The system consists of a capacitor realized by two plates, one is fixed and the other is movable but attached to a spring – see figure ($\underline{16}$).
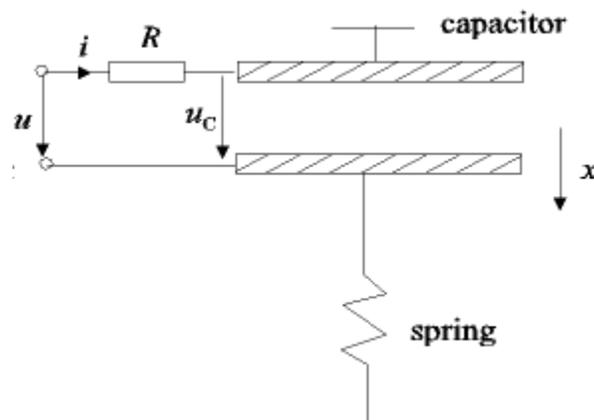
Figure 16: Example of an electromechanical system.

Further, we know that the capacitance can be described by $C(x) = k/x$. If we change the position x of the moveable plate, then a force F will take place. This force can be calculated from the equation

$$\frac{d}{dt} W^q (q, x) = \dot{c} u_C - F \frac{d}{dt} x. \tag{127}$$

In other words, the rate of energy of the capacitor is equal to the difference of the mechanical power supplied by the force and the electrical power $i_c u_c$ supplied by the electrical part. This energy relation is based on the choice q and x for the independent variables. For electrical engineers it is more common to use the voltage. Therefore, using the relation

$$\frac{d}{dt} \left(W^q (q, x) + W^u (u_C, x) \right) = \frac{d}{dt} (u_C q), \tag{128}$$

we get

$$\frac{d}{dt} (u_C q) - \frac{d}{dt} W^u (u_C, x) = i_C u_C - F \frac{d}{dt} x$$

$$u_C \frac{d}{dt} q + \left(q - \frac{\partial W^u}{\partial u_C} \right) \frac{d}{dt} u_C - \frac{\partial W^u}{\partial x} \frac{d}{dt} x = i_C u_C - F \frac{d}{dt} x \tag{129}$$

$$u_C i_C - \frac{\partial W^u}{\partial x} \frac{d}{dt} x = i_C u_C - F \frac{d}{dt} x$$

and finally

$$\frac{\partial W^u}{\partial x} = F. \tag{130}$$

Hence, the force F of the mechanical part is coupled to the electrical part by the coenergy. For the present example, it follows

$$W^u (u_C, x) = \int_0^{u_C} q\left(u_C'\right) du_C' = \int_0^{u_C} C(x) u_C' du_C' = \frac{C(x) u_C^2}{2} \tag{131}$$

and

$$F = \frac{\partial}{\partial x} W^u (u_C, x) = -\frac{k u_C^2}{2x^2}. \tag{132}$$

In principle, we can apply Newton second law for the moveable plate with mass m

$$m \frac{d}{dt} v = -\frac{k u_C^2}{2x^2} - cx, \tag{133}$$

which takes into account the law of the linear spring. On the other hand, we know that F is related to the coenergy by equation ([130](#)). This fact is a motivation for the introduction of an extended Lagrangian

$$L^{\text{ex}}(u_C, x, v) = L(x, v) + W^{\text{u}}(u_C, x) \tag{134}$$

with Lagrangian $L = T - V$ of the mechanical part.

Remark 21 *For simplicity, we have assumed that the kinetic energy is equal to the kinetic coenergy.*

For our example the equations of motion of the mechanical part are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial v}L^{\text{ex}}(u_C, x, v) - \frac{\partial}{\partial x}L^{\text{ex}}(u_C, x, v) = 0 \tag{135}$$

with

$$L^{\text{ex}}(u_C, x, v) = \frac{1}{2}mv^2 - \frac{1}{2}cx^2 + \frac{k}{2}\frac{u_C^2}{x}. \tag{136}$$

We get

$$m\frac{\mathrm{d}}{\mathrm{d}t}v = -cx - \frac{k}{2}\frac{u_C^2}{x^2}. \tag{137}$$

The equation for the electrical part (a capacitor in series with a resistor) is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial u_C}W^{\text{u}}(u_C, x) + \frac{\partial}{\partial u_C}\left(P^{\text{L}} + P^{\text{u}}\right) = 0 \tag{138}$$

with

$$W^{\text{u}}(u_C, x) = \frac{ku_C^2}{2x}, \qquad P^{\text{L}} = 0, \qquad P^{\text{u}} = \frac{(u - u_C)^2 R}{2}. \tag{139}$$

We get

$$k\frac{\mathrm{d}}{\mathrm{d}t}\frac{u_C}{x} - (u - u_C)R = \frac{k}{x}\frac{\mathrm{d}}{\mathrm{d}t}u_C - \frac{ku_C}{x^2}\frac{\mathrm{d}}{\mathrm{d}t}x - (u - u_C)R = 0. \tag{140}$$

4.2 Mechanical Forces of Electromechanical Coupling

The first step in analyzing a complicated electromechanical system by a conservation of energy approach is to reduce the system containing electromechanical coupling terms to a minimum. To do this, separate out all purely electrical parts and all purely mechanical parts of the system including losses. This separation procedure is carried out to the extent that each electrical terminal pair is coupled to one energy store, either magnetic or electrical. Any internal interconnections between circuits that are coupled to different energy storages are included in the external electrical network. The mechanical

variables represented by the mechanical terminal pairs are those which affect energy storage in the electric and magnetic fields. The separation procedure results in the general conservative electromechanical coupling network in figure 17 in which there are *n* electrical terminals and *m* mechanical terminals pairs. Each electrical terminal pair will be coupled to either a magnetic field energy storage or an electric energy field storage.
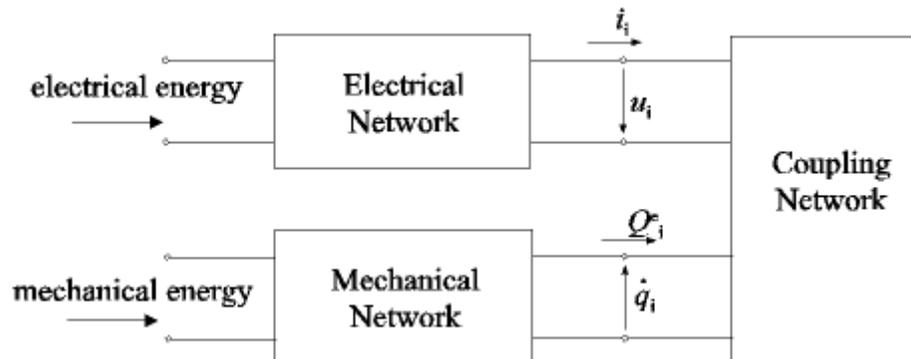
Figure 17: Simplification of electromechanical systems.

The total stored energy *W* in the coupling network is given by

$$W = W^{\bar{q}} + W^{\psi}, \tag{141}$$

where $W^{\bar{q}}$ is energy stored in electric fields and W^{ψ} is energy stored in magnetic fields. We assume that *W* is state function and given by the instantaneous configuration of the system.

Remark 22 *Hysteresis can not be taken into account. Otherwise the assumption that W is a state function would be violated.*

Consider an electrical terminal pair coupled to the electrical field storage. When the $\bar{q}_i$ and q_i are specified independently, the current in the *i*th terminal is $i_i = d\bar{q}_i/dt$ and the voltage u_i at the *i*th terminal is given by the internal constraints. Next, consider an electrical terminal pair that are coupled to magnetic field storage. When the ψ_i and x_i are specified independently, the voltage in the *i*th terminal is $u_i = d\psi_i/dt$ and the current i_i at the *i*th terminal is given by the internal constraints. It should be mentioned that instead of specifying the $\bar{q}_i$ and ψ_i the voltages u_i and the currents i_i could have been considered as independent. This is in accordance to the results obtained in section 3.

The next problem is to find the generalized force due to the electromechanical coupling. Since the *m* mechanical terminal pairs are characterized by *m* independent variables, it is possible to consider each mechanical terminal pair individually to find the force. Let us define the generalized force Q_k^e – see figure 17 – as the force applied to the *k*th mechanical coordinate by the coupling network. Q_k^e can be found by considering that an arbitrary placement dq_k of the *k*th mechanical coordinate during the time dt takes place. All other mechanical coordinates are fixed and the electrical variables may change in accordance to the internal constraints due to the electrical network. This means that only one electrical variable at each electrical terminal can be changed arbitrary. During the displacement the conservation

of energy must hold. The various energies involved in the arbitrary displacement are

- energy supplied at electrical terminals:

$$\sum_{i=1}^{n} u_i i_i \mathrm{d}t$$

- energy supplied at the *k*th mechanical terminal:

$$-Q_k^e \dot{q}_k \mathrm{d}t = -Q_k^e \mathrm{d}q_k$$

- change in stored electrical and magnetic energy of coupling field: d*W*

All lossy elements are either part of the purely electrical network or of the purely mechanical network. Hence, the conservation of energy requires that the sum of the input energy must be equal to the change in stored energy

$$\sum_{i=1}^{n} u_i i_i \mathrm{d}t - Q_k^e \mathrm{d}q_k = \mathrm{d}W. \tag{144}$$

Then, the generalized force applied to the *k*th terminal is

$$Q_k^e = \frac{\sum_{i=1}^{n} u_i i_i \mathrm{d}t - \mathrm{d}W}{\mathrm{d}q_k}. \tag{145}$$

We assume the all electrical energy storage will be in capacitances and all magnetic field storage will be in inductances. Thus, the problems of electrical and magnetic field coupling can be treated separately.

4.2.1 Mechanical Forces Due the Magnetic Field Coupling

In such cases the coupling network consists of *n* coils – see Figure 18 for an example with one coil.
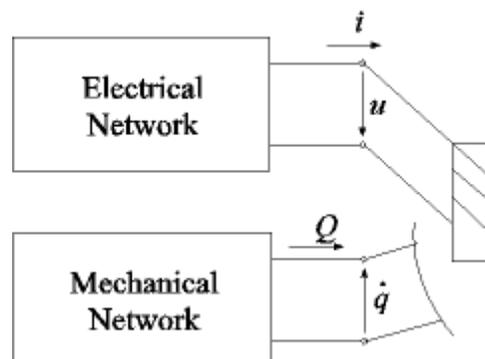
Figure 18: The coupling network is represented by a coil.

The conservation of energy establishes that the energy input from all sources is stored as magnetic field energy

$$W^\psi = W^\psi(\psi_1 \ldots, \psi_n; q_1, \ldots, q_m) = \text{input electrical energy} + \text{input mechanical energy} \qquad (146)$$

or

$$W^\psi = \int\limits_{0,\ldots,0}^{\psi_1,\ldots,\psi_n} \sum_{i=1}^{n} i_i \mathrm{d}\psi'_i + \int\limits_{0,\ldots,0}^{q_1,\ldots,q_m} \sum_{i=1}^{m} Q_i \mathrm{d}q'_i. \qquad (147)$$

which is in accordance to equation (90). The energy stored in the magnetic field coupling can be determined by bringing all system variables to their final values in an arbitrary manner. For example all flux linkages are hold at zero ($W^\psi = 0$) and the mechanical coordinates are assembled, then establish the flux linkages with the mechanical coordinates held at their final positions. For this case, we get

$$W^\psi = \int\limits_{0,\ldots,0}^{\psi_1,\ldots,\psi_n} \sum_{i=1}^{n} i_i\left(\psi'_1 \ldots, \psi'_n; q_1, \ldots, q_m\right) \mathrm{d}\psi'_i, \qquad (148)$$

where W^ψ is evaluated as the integral of $i\mathrm{d}\psi$ for any fixed q_i. Now that the stored magnetic energy has been determined, the mechanical forces due to the magnetic field coupling can be calculated. Using the relation $u_i = \mathrm{d}\psi_i/\mathrm{d}t$ and the equation (144) we get

$$Q_k^e \mathrm{d}q_k = \sum_{i=1}^{n} i_i \mathrm{d}\psi_i - \mathrm{d}W^\psi. \qquad (149)$$

Since the ψ_i, $i = 1, \ldots, n$ and the q_k are independent variables, the differentiation of $\mathrm{d}W^\psi$ yield

$$\mathrm{d}W^\psi = \frac{\partial W^\psi}{\partial q_k} \mathrm{d}q_k + \sum_{i=1}^{n} \frac{\partial W^\psi}{\partial \psi_i} \mathrm{d}\psi_i. \qquad (150)$$

All other q_i are hold constant. Taking into account equation (148) we get

$$Q_k^e \mathrm{d}q_k = \sum_{i=1}^{n} i_i \mathrm{d}\psi_i - \mathrm{d}W^\psi = \sum_{i=1}^{n} i_i \mathrm{d}\psi_i - \frac{\partial W^\psi}{\partial q_k} \mathrm{d}q_k + \sum_{i=1}^{n} i_i \mathrm{d}\psi_i = -\frac{\partial W^\psi}{\partial q_k} \mathrm{d}q_k \qquad (151)$$

and finally

$$Q_k^e = -\frac{\partial W^\psi}{\partial q_k}. \qquad (152)$$

This relation states how the generalized force applied to the kth terminal depends on the magnetic energy in terms of the flux linkages at a certain choice for the mechanical coordinates. In most cases it is preferred to express this relation in terms of the currents through the magnetic coils. Currents are usually used in the description of the electrical network. This requires the introduction of the coenergy. We start with the equation (151) and integrate it by parts

$$W^\psi = \sum_{i=1}^{n} i_i \psi_i - \int_{0,\dots,0}^{i_1,\dots,i_n} \sum_{i=1}^{n} \psi_i \left(i_1' \dots, i_n'; q_1, \dots, q_m \right) di_i', \tag{153}$$

where the second term is called the magnetic coenergy

$$W^{\mathrm{i}} = \int_{0,\dots,0}^{i_1,\dots,i_n} \sum_{i=1}^{n} \psi_i \left(i_1' \dots, i_n'; q_1, \dots, q_m \right) di_i' \tag{154}$$

– see equation ([94]). Substitution of equation ([154]) into ([152]) leads to the desired expression.

$$Q_k^e = \frac{\partial W^{\mathrm{i}}}{\partial q_k}. \tag{155}$$

The several forms of the generalized electromechanical coupling force Q_k^e applied to the kth terminal by a magnetic field as found by an arbitrary displacement of the kth mechanical coordinate q_k are summarized in Table ([156]).

Stored magnetic energy	$W^\psi = \displaystyle\int_{0,\dots,0}^{\psi_1,\dots,\psi_n} \sum_{i=1}^{n} i_i \left(\psi_1' \dots, \psi_n'; q_1, \dots, q_m \right) d\psi_i'$
Magnetic coenergy	$W^{\mathrm{i}} = \displaystyle\int_{0,\dots,0}^{i_1,\dots,i_n} \sum_{i=1}^{n} \psi_i \left(i_1' \dots, i_n'; q_1, \dots, q_m \right) di_i'$
Relation	$W^\psi + W^{\mathrm{i}} = \displaystyle\sum_{i=1}^{n} i_i \psi_i$
Independent coordinates q_i, i_i	Force: $Q_k^e = \dfrac{\partial W^{\mathrm{i}}}{\partial q_k}$
Independent coordinates q_i, ψ_i	Force: $Q_k^e = -\dfrac{\partial W^\psi}{\partial q_k}$

$$(156)$$

Remark 23 *Suppose the force $Q_k^e = \partial W^{\mathrm{i}}/\partial q_k$. This force is independent of the changes in i_i and ψ_i which take place during the arbitrary displacement. Consequently this expression is valid regardless of how i_i and ψ_i vary, if the variation is compatible with the internal constraints given by the electrical network.*

Remark 24 *From a mathematical point of view the partial derivative is taken with respect to q_k holding all other q_i and the i_i constant. The holding of i_i constant has nothing to do with electrical terminal constraints.*

Remark 25 *Suppose a electrical linear system. That means the fluxes are related to the currents via* $\psi_i = L_i i_i$. *Then we have*

$$W^{\mathrm{i}} = \int_{0,\dots,0}^{i_1,\dots,i_n} \sum_{i=1}^{n} L_i i_i' \mathrm{d}i_i' = \sum_{i=1}^{n} \frac{L_i i_i^2}{2} \tag{157}$$

and

$$W^{\psi} = \sum_{i=1}^{n} i_i \psi_i - W^{\mathrm{i}} = \sum_{i=1}^{n} i_i^2 L - \sum_{i=1}^{n} \frac{L_i i_i^2}{2} = \sum_{i=1}^{n} \frac{L_i i_i^2}{2} = W^{\mathrm{i}}. \tag{158}$$

Thus for electrical linear systems the stored magnetic energy is equal to the magnetic coenergy.

4.2.2 Mechanical Forces Due the Electrical Field Coupling

We have determined the mechanical forces produced by the magnetic field coupling. A similar development can be made for finding the mechanical forces due to the electrical coupling. In such cases the coupling network consists of *l* capacitances. The conservation of energy establishes that the energy input from all sources is stored as electric field energy

$$W^{\mathrm{q}} = W^{\mathrm{q}}(q_1 \dots, q_l; q_1, \dots, q_m) = \text{input electrical energy} + \text{input mechanical energy} \tag{159}$$

or

$$W^{\mathrm{q}} = \int_{0,\dots,0}^{q_1,\dots,q_l} \sum_{i=1}^{l} u_i \mathrm{d}q_i' + \int_{0,\dots,0}^{q_1,\dots,q_m} \sum_{i=1}^{m} Q_i \mathrm{d}q_i'. \tag{160}$$

which is in accordance to equation ([90](#)). The energy stored in the electrical field coupling can be determined as

$$W^{\mathrm{q}} = \int_{0,\dots,0}^{q_1,\dots,q_l} \sum_{i=1}^{l} u_i \left(q_1' \dots, q_l'; q_1, \dots, q_m \right) \mathrm{d}q_i', \tag{161}$$

where W^{q} is evaluated as the integral of $u \mathrm{d}q$ for any fixed q_i. For the electrical field case, just as it was for the magnetic field case, it is the interchange of energy among electrical and mechanical sources and the stored electrical energy that is a manifestation of energy conversion. This, and the fact that the stored energy is a state function which is determined by the instantaneous values of the variables, allows the use of the stored electrical energy to find the mechanical forces. Using the relation $q_i = \mathrm{d}i_i/\mathrm{d}t$ and the equation ([144](#)) we get

$$Q_k^e dq_k = \sum_{i=1}^{l} u_i d\overline{q} - dW^{\overline{q}}. \tag{162}$$

Since the $\overline{q}_i$, $i = 1, \ldots, l$ and the q_k are independent variables the differentiation of $dW^{\overline{q}}$ yield

$$dW^{\overline{q}} = \frac{\partial W^{\overline{q}}}{\partial q_k} dq_k + \sum_{i=1}^{l} \frac{\partial W^{\overline{q}}}{\partial \overline{q}} d\overline{q}. \tag{163}$$

Taking into account equation ([161](#)) we get

$$Q_k^e dq_k = \sum_{i=1}^{l} u_i d\overline{q} - dW^{\overline{q}} = \sum_{i=1}^{l} u_i d\overline{q} - \frac{\partial W^{\overline{q}}}{\partial q_k} dq_k + \sum_{i=1}^{l} u_i d\overline{q} = \frac{\partial W^{\overline{q}}}{\partial q_k} dq_k \tag{164}$$

and finally

$$Q_k^e = -\frac{\partial W^{\overline{q}}}{\partial q_k}. \tag{165}$$

This relation states how the generalized force applied to the kth terminal depends on the electrical energy in terms of the charges at a certain choice for the mechanical coordinates. In most cases it is preferred to express this relation in terms of the voltages along the capacitors. Voltages are usually used in the description of the electrical network. Like for the magnetic field case, this requires the introduction of the coenergy. We start with the equation ([160](#)) and integrate it by parts

$$W^{\overline{q}} = \sum_{i=1}^{l} \overline{q} u_i - \int_{0,\ldots,0}^{u_1,\ldots,u_l} \sum_{i=1}^{l} \overline{q} \left(u_1' \ldots, u_l'; q_1, \ldots, q_m \right) du_i', \tag{166}$$

where the second term is called the electrical coenergy

$$W^u = \int_{0,\ldots,0}^{u_1,\ldots,u_l} \sum_{i=1}^{l} \overline{q} \left(u_1' \ldots, u_l'; q_1, \ldots, q_m \right) du_i' \tag{167}$$

– see equation ([94](#)). Substitution of equation ([167](#)) into ([165](#)) leads to the desired expression.

$$Q_k^e = \frac{\partial W^u}{\partial q_k}. \tag{168}$$

The several forms of the generalized electromechanical coupling force $Q_k{}^e$ applied to the kth terminal by a electrical field as found by an arbitrary displacement of the kth mechanical coordinate q_k are summarized in Table ([169](#)).

Stored electrical energy	$W^{\bar{q}} = \displaystyle\int_{0,\dots,0}^{\bar{q}_1,\dots,\bar{q}_l} \sum_{i=1}^{l} u_i\left(\dot{q}_1',\dots,\dot{q}_l';q_1,\dots,q_m\right) d\dot{q}_i'$	
Electrical coenergy	$W^{u} = \displaystyle\int_{0,\dots,0}^{u_1,\dots,u_l} \sum_{i=1}^{l} \bar{q}_i\left(u_1',\dots,u_l';q_1,\dots,q_m\right) du_i'$	
Relation	$W^{\bar{q}} + W^{u} = \displaystyle\sum_{i=1}^{l} \bar{q}_i u_i$	(169)
Independent coordinates q_i, u_i	Force: $Q_k^e = \dfrac{\partial W^u}{\partial q_k}$	
Independent coordinates $q_i, \bar{q}_i$	Force: $Q_k^e = -\dfrac{\partial W^{\bar{q}}}{\partial q_k}$	

4.3 Equations of Motion

In the previous sections we have defined the generalized coordinates and state functions for different kinds of physical domains separately. Now, the way of modelling is given as follows

- mechanical part
  1. Select a suitable set of coordinates $\mathbf{q}^T = [q_1, \dots, q_m]$ to represent the mechanical configuration of the system.
  2. Obtain the kinetic coenergy $\overset{\cdot}{T}$ and the Rayleigh function P^R as a function of the time derivatives.
  3. If the system is conservative, find the potential energy V as a function of the coordinates, or, if the system is not conservative, find the generalized forces Q_j^e.

- electrical part:
  1. The generalized coordinates are chosen as the currents $\mathbf{i}_L^T = [i_{L,1}, \dots, i_{L,n_L}]$ through the inductances and the voltages $\mathbf{u}_C^T = [u_{C,1}, \dots, u_{C,n_C}]$ along the capacitors.
  2. Obtain the total electric coenergy

  $$W^{el} = W^u(\mathbf{u}_C, \mathbf{u}_0, \mathbf{i}_0; \mathbf{q}) + W^i(\mathbf{u}_C, \mathbf{u}_0, \mathbf{i}_0; \mathbf{q})$$

  as a function of the mechanical coordinates and the electrical coordinates and the sources $\mathbf{u}_0$, $\mathbf{i}_0$.

  3. Calculate the power quantities

  $$P^L(\mathbf{u}_C, \mathbf{u}_0, \mathbf{i}_0, \mathbf{i}_L), P^u(\mathbf{u}_C, \mathbf{u}_0, \mathbf{i}_0, \mathbf{i}_L), P^C(\mathbf{u}_C, \mathbf{u}_0, \mathbf{i}_0, \mathbf{i}_L), P^i(\mathbf{u}_C, \mathbf{u}_0, \mathbf{i}_0, \mathbf{i}_L)$$

- Define the extended Lagrangian

$$L^{\mathrm{ex}} = T' + W^{\mathrm{el}} - V.$$

Then, the equations of motion of the mechanical part are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} q_i = \dot{q}_i, \qquad \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_i} L^{\mathrm{ex}} - \frac{\partial}{\partial q_i} L^{\mathrm{ex}} + \frac{\partial}{\partial \dot{q}_i} P^{\mathrm{R}} = Q_i^{\mathrm{e}}, \qquad i = 1, \ldots, m$$

and the equations for the electrical part are

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial u_{C,k}} W^{\mathrm{el}} + \frac{\partial}{\partial u_{C,k}} \left(P^{\mathrm{L}} + P^{\mathrm{u}} \right) = 0, \qquad k \in \mathcal{C}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial i_{L,k}} W^{\mathrm{el}} + \frac{\partial}{\partial i_{L,k}} \left(P^{\mathrm{C}} + P^{\mathrm{i}} \right) = 0, \qquad i \in \mathcal{L}.$$

4.4 Electrical Drives

4.4.1 Elementary Machine

The purpose of this section is to derive the magnetic coenergy of an elementary machine. This will act as starting point for the considerations on DC–drives. Figure 19 presents an elementary two pole machine with one winding on the stator and one on the rotor.
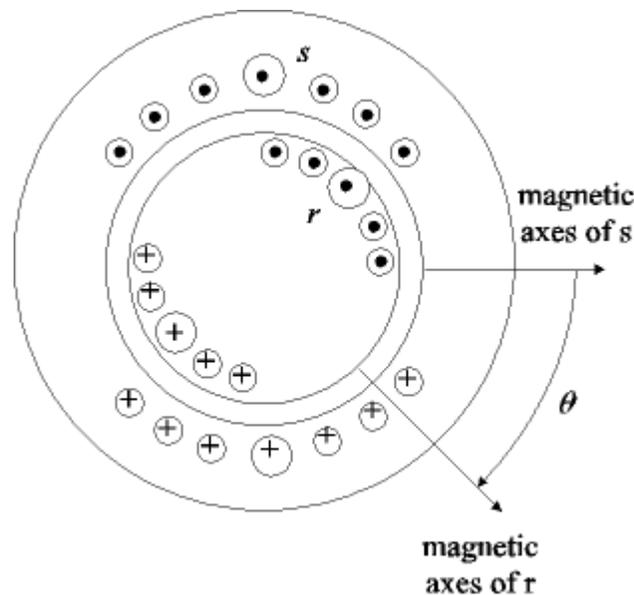
Figure 19: Elementary machine.

These windings are distributed over a number of slots so that their magnetomotive force can be approximated by space sinusoids. The stator and rotor are concentric cylinders, and slot openings are neglected. On these assumptions the stator and rotor self–inductances L_{ss} and L_{rr} are constant. The stator–rotor mutual inductance depends on the angle θ between the magnetic axes of the stator and rotor

windings. The mutual inductance is a positive maximum when $\theta = 0(2\pi)$, is zero when $\theta = \pm\pi/2$, and is a negative maximum when $\theta = \pm\pi$. On the assumptions of sinusoidal waves and a uniform air gap, the space–wise distribution of the air–gap flux in sinusoidal, and the mutual inductance is

$$L(\theta) = L_{sr}\cos(\theta). \tag{175}$$

L_{sr} is the value of L when the magnetic axes of the rotor and the stator are aligned. In a linear system, the relationship between the fluxes and currents are given with

$$\psi_s = L_{ss}i_s + L(\theta)i_r, \qquad \psi_r = L_{rr}i_r + L(\theta)i_s. \tag{176}$$

The coenergy in the magnetic field in the air gap is given by

$$dW^i(i_s, i_r, \theta) = \psi_s di_s + \psi_r di_r + M d\theta \tag{177}$$

and we have

$$
\begin{aligned}
\int dW^i(i_s, i_r, \theta) &= \int_0^{i_r} \psi_r\left(i_s = 0, i'_r, \theta\right) di'_r + \int_0^{i_s} \psi_s\left(i_r = \text{constant}, i'_s, \theta\right) di'_s \\
&= \int_0^{i_r} L_{rr}i_r di'_r + \int_0^{i_s} (L_{ss}i_s + L(\theta)i_r) di'_s \\
&= \frac{L_{rr}i_r^2}{2} + \frac{L_{ss}i_s^2}{2} + L(\theta)i_r i_s \\
&= \frac{L_{rr}i_r^2}{2} + \frac{L_{ss}i_s^2}{2} + L_{sr}\cos(\theta)i_r i_s
\end{aligned}
\tag{178}
$$

4.4.2 DC–drive

For our purpose it is suffices to say that a DC–drive is an elementary machine with commutator. The task of the rotating commutator is to convert the AC– voltage generated in each rotating armature coil to DC in the external armature terminals by means of the stationary brushes to which the armature leads are connected. Moreover, the magnetic axes of the armature winding is perpendicular to the magnetic axes of the field winding. See [2] for a detailed introduction in the theory and application of DC–drives.

For convenience we assume a sinusoidal flux density wave in the air gap. Then, we apply the coenergy equation (178)

$$W^i(i_E, i_A, \theta) = \frac{L_A i_A^2}{2} + \frac{L_E i_E^2}{2} \ \square \ c\cos(\theta) i_A i_E, \tag{179}$$

with the external armature current i_A, the exciting current i_E, the angle θ between the magnetic axes of rotor and stator, the inductances L_A and L_E, and the coupling constant c. For DC–generators the plus sign has to be taken, the minus sign is related to a DC–motor. The losses in the windings are taken into account using the resistors R_A and R_E. It follows the torque with

$$M = \frac{\partial W^i (i_E, i_A, \theta)}{\partial \theta} = \mp c \sin (\theta) i_E i_A. \tag{180}$$

Next the process of commutation has to be taken into account. The commutator ensures that the angle θ between the air–gap flux and the armature magnetomotive force is 90° electrical degrees. Hence, the commutation leads to

$$M = \mp c i_E i_A. \tag{181}$$

The Figure 20 shows a schematic representation of the separate excited DC–drive.
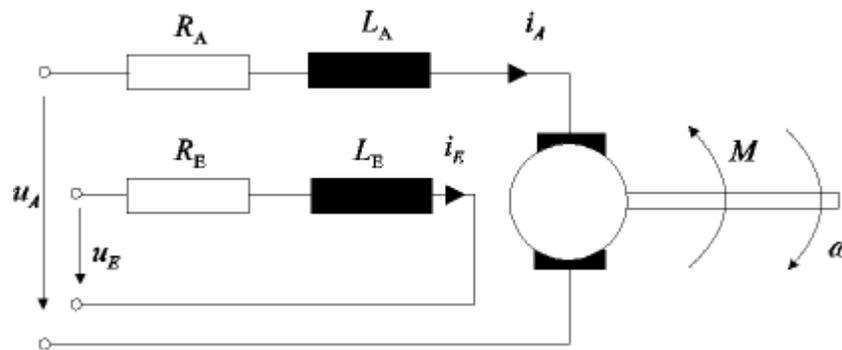
Figure 20: Separate excited DC-drive.

The derivation of the equations of motion follows the following procedure using the coenergy of the magnetic field of a DC–drive before commutation – see equation (179):

- *Mechanical part and armature current part:* The derivatives with respect to the coordinates θ, i_E and the time have to be carried out. Finally, the commutator condition requires $\sin (\theta) = 1$.
- *Exciting current part:* First, the commutator condition requires $\sin (\theta) = 1$. Then, the derivatives with respect to the coordinates θ, i_E and the time have to be carried out.

Separate excited DC—drive: Next, this procedure is applied to a simple DC–drive with load. The extended Lagrangian is the sum of the mechanical Lagrangian and the magnetic coenergy (coordinates i_A, i_E)

$$L^{ex} (i_A, i_E, \theta, \omega) = L (\theta, \omega) + W^i (i_A, i_E, \theta). \tag{182}$$

At this state the mechanical rotor angle φ and the angle θ between the magnetic axes of rotor and stator are the same. This leads for a DC–motor

$$L^{ex} (i_A, i_E, \theta, \omega) = \frac{J}{2} \omega^2 + \frac{L_A i_A^2}{2} + \frac{L_E i_E^2}{2} - c \cos (\theta) i_A i_E \tag{183}$$

with the inertia *J*. Next, we apply the relations (173) and get

$$J\dot{\omega} - c\sin(\theta)i_A i_E = -M_L \tag{184}$$

with the external load M_L. The commutation requires $\sin(\theta) = 1$ which gives

$$\frac{d}{dt}\varphi = \omega, \qquad J\dot{\omega} = c i_A i_E - M_L. \tag{185}$$

Here, the rotor angle φ is the generalized coordinate of the movement and φ is different from θ. We assume constant excitement – that means i_E = constant – for the derivation of the equations for the electrical part. Hence, we have

$$\frac{d}{dt}\frac{\partial}{\partial i_L}W^{\text{el}} + \frac{\partial}{\partial i_L}\left(P^C + P^i\right) = 0 \tag{186}$$

with

$$P^C = 0, \qquad P^i = \frac{i_A^2 R_A}{2} + \frac{i_E^2 R_E}{2} - u_A i_A - u_E i_E. \tag{187}$$

Finally, we get

$$L_A\frac{d}{dt}i_A - c\cos(\theta)\frac{d}{dt}i_E + c i_E \sin(\theta)\omega + i_A R_A - u_A = 0 \tag{188}$$

and the commutator conditions $\theta = \pi/2$ leads to

$$L_A\frac{d}{dt}i_A + c i_E \omega + i_A R - u_A = 0 \tag{189}$$

Shunt DC—drive and Series DC—drive: Other DC–drives of interest are the shunt DC–drive with $u_A = u_E$.
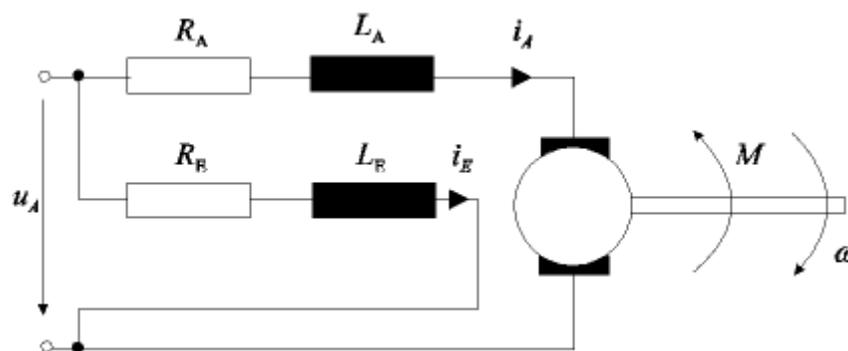
Figure 21: Shunt DC-drive.

This has no influence on the magnetic coenergy, thus equation (179) is still in force. A schematic figure of the series DC–drive is shown in figure 22. The condition $i_A = i_E$ is in force, which gives the magnetic

coenergy – see equation ([179](#))

$$W^i = \square c \cos(\theta) i_A^2 + \left(\frac{L_A}{2} + \frac{L_E}{2} \right) i_A^2.$$
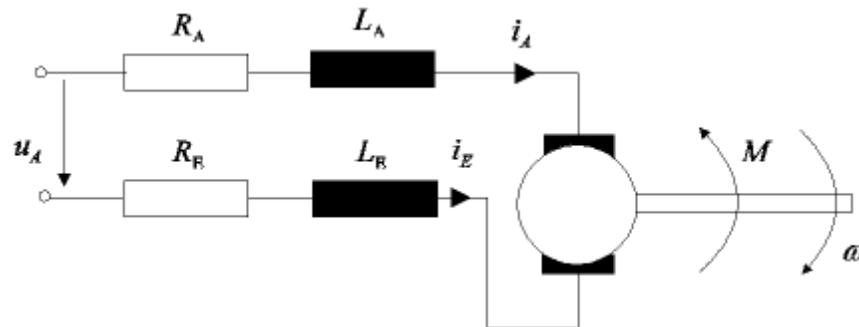
Figure 22: Series DC–drive.

4.5 Case Studies

4.5.1 Ward–Leonard drive

Modelling and Simulation

4.5.2 Ball in a Magnetic Field

Modelling and Simulation

4.5.3 Electromagnet

Modelling and Simulation

4.5.4 Relay Device

The relay shown in figure [23](#) is made from infinitely permeable magnetic material with a moveable plunger, also of infinitely permeable material.
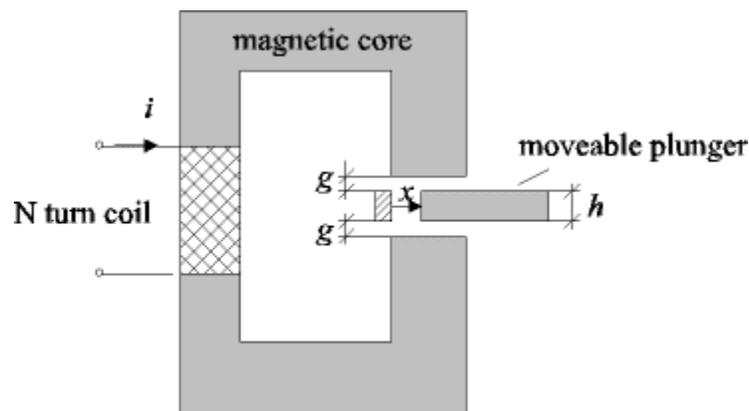
Figure 23: Relay with moveable plunger.

The height of the plunger is much greater than the air–gap length ($h \gg g$). The magnetic coenergy is defined by

$$W^i(i, x) = \int_0^i \psi(i', x)\, di' = \int_0^i L(x) i'\, di' = L(x) \frac{i^2}{2}.$$

Thus, the calculation of $L(x)$ is required. Because of the high permeability the flux is confined almost entirely in the core. The relationship between the magnetomotive force iN (N is the number of turns) and the magnetic field intensity in the core H_c, in the plunger H_p, and the gap H_g is given by

$$iN = H_c l_c + H_p h + H_g 2g$$

with the mean core length l_c. Further, the general relationship between B and H is given by

$$B = \Box H.$$

This leads to

$$iN = \frac{B_c}{\Box_c} l_c + \frac{B_p}{\Box_c} h + \frac{B_g}{\Box_0} 2g$$

with the flux density B_c in the core (uniform distributed over a cross section area $A_c = ld$ of the core), the flux density B_g in the air gap (uniform distributed over a cross section area $A_g = l(d - x)$ of the air gap), and the flux density B_p in the plunger. The field follows the path defined by the core, thus the relation

$$\phi = A_c B_c$$

holds. In the air gap and in the plunger the flux ϕ is almost the same as in the core. Hence,

$$\phi = A_g B_g = A_p B_p.$$

This gives

$$iN = \left(\frac{l_c}{\Box_c A_c} + \frac{h}{\Box_c A_g} + \frac{2g}{\Box_0 A_g} \right) \phi$$

and

$$L(x) = \frac{\psi}{i} = \frac{N\phi}{i} = \frac{N^2}{\left(\frac{l_c}{\Box_c A_c} + \frac{h}{\Box_c A_g} + \frac{2g}{\Box_0 A_g} \right)} \approx \Box_0 N^2 \frac{A_g}{g}.$$

The air gap cross section area is

$$A_g = l(d - x)$$

which gives

$$L\left(x\right) = \square_0 N^2 \frac{ld}{2g}\left(1 - \frac{x}{d}\right).$$

The magnetic energy follows with

$$W^{\mathrm{i}}\left(i, x\right) = \square_0 N^2 \frac{ld}{4g}\left(1 - \frac{x}{d}\right)i^2$$

and the related force with

$$F = \frac{\partial W^{\mathrm{i}}\left(i, x\right)}{\partial x} = -\square_0 N^2 \frac{l}{4g}i^2.$$

References

[1]   F. Cellier: Continuous System Modelling, Springer Verlag, 1991.

[2]   A. Fitzgerald, C. Kingsley, S. Umans, Electric Machinery, McGraw–Hill, 1990.

[3]   G. Folwes: Analytical Mechanics, Saunders, 1982.

[4]   H. Goldstein: Klassische Mechanik, AULA Verlag, 1991.

[5]   M. Hirsch, S. Smale: Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, 1974.

[4]   A. Kugi, K. Schlacher: Nonlinear H_∞–Controller Design for a DC-to-DC Power Converter, IEEE Transactions on Control Systems Technology , in press.

[6]   J. Shearer, B. Kulakowski: Dynamic Modeling and Control of Engineering Systems, Mc Millan, 1990.

[7]   K. Schlacher, A. Kugi, R. Scheidl: Tensor Analysis Based Symbolic Computation for Mechatronic Systems, Mathematics and Computers in Simulation, Vol. 46, 517-525, 1998.

[8]   A. Zemanian: Transfinitness for Graphs, Electrical Networks, and Random Walks, Birkhäuser, 1996.